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ASYMPTOTIC METHODS IN THE THEORY OF NONLINEAR OSCILLATIONS

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NOTE

This book is devoted to the approximate asymptotic methods of solving the problems in the theory of nonlinear oscillations met in many fields of physics and engineering. It is intended for the wide circle of engineering-technical and scientific workers who are concerned with oscillatory processes.

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PREFACE

At the present time the questions of nonlinear oscillations are attracting great attention in widely varying fields of engineering and physics.

The methods of asymptotic expansion in powers of a small parameter are ver fective means for studying nonlinear oscillations.

By their aid, in a large number of cases of practical importance, it is to obtain relatively simple computation layouts and detailed interpretation o character of the course of the oscillatory process.

In this connection, there is a definite need for a book describing this odology in the simplest possible form, without requiring an excessive math background of the reader.

The book by N.M.Krylov and N.N.Bogolyubov, "Introduction to Nonlinear ics" (Bibl.20), published in 1937 and devoted to precisely these questions, bibliographic scarcity today; the methods worked out by these authors have siderably expanded.

In this sense, the present book is submitted to the reader's scruti

Its basic purpose is to describe the method of asymptotic expansion of a small parameter in its modern form, with respect to the problems of mechanics.

For this reason, the examples given are mainly of an illustrative the book makes no claim whatsoever to complete coverage of the theory oscillations nor of the physical phenomena considered in it.

The book consists of an introduction and five chapters.

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In Chapter 1, natural oscillations in quasi-linear systems with one degree of freedom are discussed.

Chapter 2 contains basic elementary information on the method of the phase plane. Free oscillations in relaxation type systems [illegible] also discussed.

For the understanding of the question of the transition to the discontinuous treatment of relaxation oscillations, we are giving the fundamental propositions of the method of the large parameter developed by A.A.Dorodnitsyn.

Chapter 3 is devoted to a study of oscillatory systems under the influence of external periodic forces.

Chapter 4 describes the methods of the mean, by the aid of which systems with many degrees of freedom can be considered.

These four chapters are written for a reader familiar with mathematics to the extent covered by the normal course of Polytechnic Institutes.

Chapter 5 is intended for mathematicians who are interested in questions of the theory of differential equations with a small parameter. In it questions of the justification of asymptotic methods are discussed and a series of theorems on the existance and stability of periodic and quasi-periodic solutions are established.

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INTRODUCTION

1. The study of oscillatory processes is of fundamental importance for the widely varying branches of mechanics, physics, and engineering. The vibrations of structures and machines, the electromagnetic oscillations in radio engineering and optics, self-sustained oscillations in systems of automatic control and servosyste sonic and ultrasonic oscillations - all these oscillatory processes, seemingly dif- ferent, with no resemblance whatever to each other, are correlated by the methods mathematical physics into one general doctrine of oscillations.

It should be noted that, as science and engineering develop, the role of the doctrine of oscillations is also rapidly expanding. Disregarding such disciplines as radio engineering and acoustics which have been completely "covered" by the doc trine of oscillations, let us take something like machine building as a typical ex ample. It is not so long ago that no particular importance was attached in this field to the study of oscillations, and stress calculations were based on static concepts on the relation between deformation and load. However, with the tenden toward increasing rotational speeds and decreasing dimensions, the role of oscill tions in the transition to high-speed machine building can no longer be disregar The numerous accidents, due to the increased actual loads produced by the excit of oscillations, have made it imperative for designers and engineers to study, care, the possible vibrations of machine units, and to estimate their intensity

The sources of the modern theory of oscillation are clearly defined in the classical mechanics of the times of Galileo, Huygens, Newton, in the problem motion of the pendulum. The works of Lagrange contain already formulations of theory of small oscillations. In its further development, the term "theory o

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linear oscillations" was used, i.e., oscillations characterized by linear differential equations with constant coefficients, with either homogeneous or free terms, being known functions of time.

In the work of many scientists, linear differential equations have been an efficient tool of research. Thus, A.N.Krylov and his students, developing the theory of linear oscillations, applied it successfully to the solution of the problem of roll of a ship, to the theory of the gyroscope, and to artillery problems.

The simplicity of the basic principles of the theory of linear differential equations with constant coefficients has resulted in a large amount of work done on the theory of linear oscillations, in a feasible generality for formulating its la[w] and in clearness of its physical interpretation. Fundamental concepts of this theory, such as natural frequency, damping decrement, resonance, normal vibrators, et[c.] have enjoyed wide popularity and have been an irreplacible means of investigation [i]n almost all branches of physics and engineering. The property of linearity of differentiating operators, interpreted as the principle of superposition of oscillations, the fact of the transition of harmonic functions of time (when these operators are applied) into harmonic functions of the same frequency, have permitted in[-] vestigating the influence of arbitrarily applied forces on the linear oscillatory system to be reduced to the investigation of the influence of forces of the simpl[e] type, harmonically dependent on time. Thereby the "spectral" approach to the osc[il]latory processes - an approach that has had immense importance even outside the [the]ory of oscillations in the proper sense of the word - has been worked out.

The technique of calculating specific linear oscillatory systems, under the stimulating influence of electrical engineering, has been enriched by the creati[on] of the so-called symbolic method and of various modifications of it, for example the method of complex amplitudes. Its basic idea is that, since the differenti[al] operator, in combination with the constant coefficients, obeys the very same dis[-] tributive, associative, and commutative laws as ordinary numbers do, it follow[s] the differentiating operator with respect to time may be replaced by a certain [sym]bol and the systems of linear differential equations with constant coefficient[s] be, formally, reduced to linear algebraic equations. By solving them, we obt[ain]

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"symbolic solution", which must then be deciphered, using a certain system of operations.

In the case of oscillatory systems with an infinite number of degrees of freedom (or, as they are also called, systems with distributed parameters), which are described by partial differential equations, which, in addition to differentiation with respect to time, also include differentiation with respect to other independe variables, the symbolic method leads to equations with a smaller number of variabl which already represents a very substantial simplification.

After the fundamental work of Heaviside, the symbolic method began to be successfully used, mainly in electrical engineering, for the solution of numerous pro lems. However, for a long time mathematicians doubted its legitimacy and justific tion. It is only since the 1920's, after the work of Carson, Deutch, Bromwich, an others, that the mathematical aspect of the symbolic method first started to beco relatively clear by being linked to the Laplace transformations and to the powerf methods of the theory of functions of a complex variable.

An extensive literature is today devoted to the theory and application of th symbolic methods.

In the USSR, work in the field of the symbolic method has been done by A.M.Efros and A.M.Danilevskiy, by N.M.Krylov and N.N.Bogolyubov, by A.I.Lur'ye, others.

In view of the fact that the theory of linear oscillations, for the abovementioned reasons, has been developed in very great detail, and in view of the that its mathematical apparatus functions, one might say, almost automatically, vestigators have attempted to classify oscillatory processes studied by them i linear categories as far as possible, frequently discarding the nonlinear term without proper justification. In this case, they sometimes completely lost si of the fact that such a "linear" treatment might lead to substantial errors, merely of the quantitative, but even of fundamental and qualitative types.

During the initial stage of development of the doctrine of oscillations, was only in isolated cases that linearization was not used and that nonlinear

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cillations were actually considered as such (Ostrogradskiy, Helmholtz, Rayleigh). And yet it must be emphasized that, even in the 19th Century, there already existed a mathematical apparatus which, on proper development and generalization, could have been applied to the study of nonlinear oscillations, sufficiently close to linear. Oscillations for which the corresponding differential equations, although they are nonlinear, still contain some parameter ε, entering these equations so that at zero value of ε they degenerate into linear differential equations with constant coefficients, are usually termed sufficiently-close-to-linear. It is assumed in this cas that the parameter ε is "small", i.e., that it may be taken sufficiently small in absolute value. Speaking of such a mathematical apparatus, we are thinking primari of the theory of perturbations, developed by astronomers for studying the motions the planets. Here, too, we must deal with the study of motions described by differential equations containing a small parameter. When this parameter becomes ze they degenerate into equations integrable by elementary methods, usually into th equations of "the problem of two bodies". This type of problem, in particular th famous "problem of three bodies", which was considered at the very beginning of lestial mechanics, soon disclosed a substantial difficulty which consisted in th impossibility of utilizing the ordinary expansions in powers of a small paramete obtain results suitable for studying the motion over a sufficiently long time in val.

The point is that the usual expansions in power series of a small paramete lead, for the quantities sought, characterizing the motion, to approximate for which, besides terms depending harmonically on time, also contain the so-calle ular terms of the type

$$t^m \sin \alpha t, \quad t^m \cos \alpha t, \qquad (1$$

in which the time t is outside the sine or cosine symbol. Owing to the fact the intensity of the secular terms rapidly increases with t, it is clear, eve out a detailed analysis of the error, that the field of applicability of the proximate formulas so obtained is limited to too short an interval of time.

This difficulty can be best illustrated by the trivial example of the

motion described by the equation

$$\frac{dx}{dt} = -\varepsilon x \tag{2}$$

with the small positive parameter ε. The solution of this equation is

$$x = Ce^{-\varepsilon t}. \tag{3}$$

However, if we used the usual method of expansion in power series of ε to solve this equation, we would get

$$x = C\left(1 - \varepsilon t + \frac{\varepsilon^2 t^2}{2} - \dots\right). \tag{4}$$

If the series is terminated at the first, second, or third term, i.e., if the formu- las of the first, second, etc. approximation are used, these will not show that the quantity in question is damped with increasing t, since these formulas will be ap- plicable only when $t \ll \frac{1}{\varepsilon}$, while, during this time, x cannot vary appreciably.

This property of the ordinary expansions in power series of a small paramete[r] becomes quite clear on considering the method proposed by Poisson in investigatin[g] the problem of oscillations of a pendulum.

Poisson's method consists in the following: Let it be required to find the [solu-] tion of the above-mentioned nonlinear equation containing the small parameter ε, which we may represent in the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}\right).$$

Then the solution satisfying the equation, with an accuracy to quantities of ε^{n+1} order of smallness will be obtained in the form of the series

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots + \varepsilon^n x_n.$$

On substituting the series of eq.(6) in the left-hand side of eq.(5), [the re-] sult of the substitution is expanded in powers of ε, and the terms contain[ing] a power above the n^{th} are rejected. The coefficients of equal powers of t[he parame-] ter ε are then equated.

In this way, we obtain the system of equations

$$\frac{d^2x_0}{dt^2} + \omega^2 x_0 = 0,$$

$$\left.\begin{aligned} &\frac{d^2x_1}{dt^2} + \omega^2 x_1 = f\left(x_0, \frac{dx_0}{dt}\right), \\ &\frac{d^2x_2}{dt^2} + \omega^2 x_2 = f'_x\left(x_0, \frac{dx_0}{dt}\right)x_1 + f'_{x'}\left(x_0, \frac{dx_0}{dt}\right)x'_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right| \quad (7)$$

It is easy to see, however, that the use of this method leads to the appearance of the above-mentioned secular terms in the solution.

In fact, consider the concrete equation

$$m\frac{d^2x}{dt^2} + \alpha x + \gamma x^3 = 0, \quad \alpha > 0, \quad \gamma > 0, \qquad (8)$$

which may be integrated as the equation of the undamped oscillations of a certain mass m attracted toward the position of equilibrium by a restoring elastic force:

$$p(x) = \alpha x + \gamma x^3. \qquad (9$$

Let us assume that the characteristic of the restoring force p(x) is close to linear.

Then, denoting $\frac{\alpha}{m} = \omega^2$, $\frac{\gamma}{m} = \varepsilon$, we set up the approximate solution with an accuracy to quantities of the second order of smallness.

We have:

$$x = x_0 + \varepsilon x_1,$$

where

$$\frac{d^2x_0}{dt^2} + \omega^2 x_0 = 0,$$

$$\frac{d^2x_1}{dt^2} + \omega^2 x_1 = -x_0^3.$$

From eq.(11) we find:

$$x_0 = a\cos(\omega t + \vartheta)$$

and, substituting in the right-hand side of eq.(12), we get:

$$\frac{d^2x_1}{dt^2} + \omega^2 x_1 = -\frac{3}{4}a^3\cos(\omega t + \vartheta) - \frac{1}{4}a^3\cos 3(\omega t + \vartheta)$$

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Whence we find the following value for x_1:

$$x_1 = -\frac{3}{8\omega} t a^3 \sin(\omega t+\theta) + \frac{a^3}{32\omega^2}\cos 3(\omega t+\theta). \tag{15}$$

By substituting eq.(13) and eq.(15) in eq.(10), we obtain the required solution in the form

$$x = a\cos(\omega t+\theta) - \frac{3\varepsilon}{8\omega} a^3 t \sin(\omega t+\theta) + \frac{\varepsilon a^3}{32\omega^2}\cos 3(\omega t+\theta). \tag{16}$$

The approximate solution so found contains the secular term

$$-\frac{3\varepsilon}{8\omega} a^3 t \sin(\omega t+\theta),$$

and therefore the oscillations represented by eq.(16) should expand, while their amplitude, as t increases without limit, should also increase without limit, which is in obvious contradiction with the character of the exact solution of eq.(8), which as is well known, is expressed by elliptic functions and has the following form:

$$x = x_{max}\,\mathrm{cn}\left\{\frac{2K}{\pi}\psi\right\}, \tag{17}$$

where cn, K denote respectively an elliptic cosine and a total elliptic integral of the first kind.

The failure of eq.(16) to correspond to reality is further confirmed by the following fact:

If we multiply eq.(8) by $\frac{dx}{dt}$ and integrate, we easily find the first integral, namely the integral of kinetic energy

$$\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\alpha x^2 + \frac{\gamma}{4} x^4 = E, \tag{18}$$

expressing the law of conservation of energy.

It follows from eq.(18) that for $\alpha > 0$, $\gamma > 0$, x^2 cannot be greater than $\frac{2E}{\alpha}$, and, consequently, the amplitude of the oscillations cannot increase without limit.

On analyzing these simple examples, it becomes obvious that this method of obtaining the approximate solutions by expanding x in power series of the small parameter ε is suitable only for a very short time interval.

The series of eq.(16), owing to the presence of secular terms, is not only un-

suitable for a quantitative analysis of the behavior of the solution of eq.(8) over the entire real axis, but likewise for a qualitative analysis of that behavior, even where the series of eq.(16) is convergent [cf. for instance, the first example, eq.(2)].

We remark once again that the presence of secular terms in the expansion of eq.(16) does not signify in any case that eq.(8) has no periodic solutions at all, but merely indicates the inappropriate choice of the solution.

This may be illustrated once again by the following simple example: Let us consider the function

$$\sin(\omega + \varepsilon)t, \tag{19}$$

which has the period $\frac{2\pi}{\omega + \varepsilon}$. For small values of ε and any values of ω and t, we may expand it into the series

$$\sin(\omega + \varepsilon)t = \sin\omega t + \varepsilon t\cos\omega t - \frac{\varepsilon^2 t^2}{2!}\sin\omega t - \frac{\varepsilon^3 t^3}{3!}\cos\omega t + \dots \tag{20}$$

On considering the right-hand side of eq.(20), it is difficult to establish its periodicity, in view of the presence of secular terms.

The above-mentioned difficulty of the secular terms in the theory of perturbations is of entirely the same character. A large number of effective methods of overcoming it were proposed after the work of Lagrange and Laplace. It is true that the power series in powers of a small parameter, to which they lead, are as a rule divergent, but nevertheless the approximate formulas here obtained, if we limit ourselves to a certain fixed number of terms $m = 1, 2, 3, \dots$, are very advantageous for practical calculations. The point is that these series are asymptotic in the sense that the error of the m^{th} approximation is proportional to the $(m + 1)^{th}$ power of the small parameter ε. For this reason, for a fixed value of $m = 1, 2, 3, \dots$, the error will become as small as any assigned quantity, at sufficiently small values of ε. Of course, by increasing m without limit, we will generally not obtain convergence, because of the fact that $m \to \infty$; however, the lack of this convergence is not of major importance for practical calculations, since, in practice, the de-

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termination of the coefficients of the following powers of ε becomes so complex that actually only the approximations of the first or second order (or, in general, of very low order) can be used and that their applicability is completely dependent on the asymptotic property.

The above-mentioned asymptotic methods were found to be very effective in celestial mechanics, and were later applied to quantum mechanics. It must, however, be emphasized that these methods were developed especially for conservative dynamic systems described by canonical equations, and cannot be applied without fundamental generalization to the study of most of the nonlinear oscillatory systems considered, since these systems are nonconservative, containing sources both of energy inflow and energy absorption.

In addition to the system of the theory of perturbations, a mathematical syst not specifically connected with conservative systems has been developed. Here, fir of all, we must point to the theory of linear differential equations with periodic coefficients founded by A.M.Lyapunov, and to the Lyapunov-Poincaré local theory of periodic solutions. This latter theory considers general nonlinear differential sy tems containing the small parameter ε in such a way that for ε = 0 they possess pe odic solutions, and it establishes explicit criteria for the existence and stabili of periodic solutions for sufficiently small value of ε ≠ 0. The methods of Lyap and Poincaré have the substantial advantage over the usual methods of the theory o perturbations that they are rigorously-based methods, suitable not only for quanti tative investigations but also for qualitative studies.

As will be seen from the above, the mathematical system which could be appli to the study of nonlinear oscillations already existed. It was, however, not sys tematically utilized in this field, before about the beginning of the 1930's. Its internal correlation with the problems of nonlinear oscillations had likewise not been discovered.

The Lyapunov-Poincaré methods were first applied to the systematic investig of nonlinear oscillations in 1929 by the Soviet school of physicists which is as ciated with the names of L.I.Mandel'shtam, N.D.Papaleksi, A.A.Andronov, and

It must be noted here that nonlinear oscillations themselves became particularly timely and began to evoke intensified interest only after the Twenties, in connection with the rapid development of radio engineering after the invention of the electron tube. Such problems as those of the stable generation of undamped oscillations, frequency conversion, forced synchronization, modulation, etc. could be solve only by the introduction of nonlinear elements into the oscillatory systems since, in purely linear oscillatory systems, no steady oscillatory states independent of the initial conditions can exist; under the action of external harmonic forces with a certain frequency ω, forced oscillations are excited only with the same frequency ω, etc. The electron tube proved to be an exceedingly flexible and convenie means of producing the corresponding nonlinear elements.

It was only after the appearance of numerous studies connected with problems o this type, that the profound and fundamental difference between the mechanics of no linear oscillations and the mechanics of linear oscillations first became physicall clear - a difference that is completely preserved even when we consider the weakly nonlinear oscillations described by differential equations distinguished from line differential equations with constant coefficients only by the presence of very smal terms.

Imagine a system so close to linear that its oscillations, during the course a single period, have a form very close to harmonic. If, however, we consider the oscillations over a time interval long in comparison with the period of oscillatio then the influence even of small deviations of the system from linear, expressed b the presence of small nonlinear terms in the differential equations, will manifest itself noticeably.

For example, there may be sources and sinks of energy in the system, which pr duce and absorb a very small quantity of work during each period of oscillation b whose effects, on prolonged action, may become cumulative and exert a noticeable fluence on the course of the oscillatory process, on its damping, its oscillatio and its stability. By analogy, the nonlinearity of a quasi-elastic force will e an influence, on prolonged action, on the phase of the oscillations, etc.

Thus small nonlinear terms may exert, as it were, a cumulative action.

We may also emphasize that, because of the nonlinearity, the principle of superposition is violated, and individual harmonics of the oscillations start interacting, thus making any individual consideration of the behavior of each harmonic component of the oscillations impossible.

It is entirely natural that the oscillatory systems of slight nonlinearity should be most amenable to study, since the methods of the theory of perturbations may be applied to them in one form or another.

However, the study of a system of extensive nonlinearity is a very difficult problem from the mathematical point of view, and requires an individual approach to each specific case.

The oscillatory systems with one degree of freedom, under the influence of tim independent forces, have been more or less well studied. Even this, however, covered the qualitative aspects alone.

However, for weakly nonlinear systems, described by the above differential equ tions with a small parameter and nonlinear terms, we already have today a large num ber of rather general methods, applicable to several typical classes of oscillatory systems which are often met in practice.

The Van der Pol method has been one of the earliest. In his studies Van der considered mainly equations of the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}\right) \tag{21}$$

with a small positive parameter ε. In this case, it was ordinarily assumed (the Van der Pol equation) that:

$$f\left(x, \frac{dx}{dt}\right) = (1 - x^2)\frac{dx}{dt}. \tag{22}$$

With a certain schematization, this equation, at least from the qualitative poin view, correctly describes the processes of self-excited oscillations in electro oscillators.

To obtain the first approximation, Van der Pol proposed a special method

"slowly varying" coefficients, analogous to one of the methods used as far back as Lagrange in celestial mechanics. He represented the true solution in the form of a function expressing the harmonic oscillations

$$x = a\cos(\omega t + \varphi) \tag{23}$$

with a slowly varying amplitude a and a phase φ. These latter quantities must be found from the differential equations with separable variables

$$\frac{da}{dt} = \varepsilon A(a), \quad \frac{d\varphi}{dt} = \varepsilon B(a), \tag{24}$$

where A(a), B(a) are certain functions of the amplitude defined simply in terms of the assigned expression $f(x, \frac{dx}{dt})$. By his method, Van der Pol obtained a number of important results. For example, he investigated the process of the build-up of os cillations, stationary states, oscillatory hysteresis, etc.

It must, however, be emphasized that, in Van der Pol's formulation, the appr mation was derived by purely intuitive reasoning, and although this approximation did prove to be fruitful in the first period of work in the field of nonlinear mechanics, it could not completely satis the requirements of practice. Further than that, the questi of its theoretical foundation, of the limits of its applica ty, and of the derivation of higher approximations, all re mained unanswered.

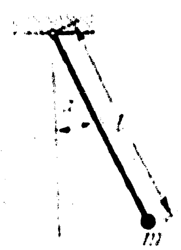

Fig.1

The basic purpose of the present book is to present an position of the asymptotic methods of nonlinear mechanics veloped by N.M.Krylov and N.N.Bogolyubov. For the study of systems with slowly varying parameters, the method of Yu.A.Mitropol'skiy is giv Our attention has been focused primarily on weakly nonlinear systems with one of freedom. At the end of the book, certain of these methods are extended to general cases.

As already stated, the systems studied by us are of a form very often met practice.

Let us therefore consider a number of typical examples of such systems.

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examples will be used later for illustrating the methods described.

2. The simplest example of a nonlinear oscillatory system is the ordinary mathematical pendulum (Fig.1). If friction is neglected, then the equation describing the oscillation of the pendulum has the form

$$ml^2\frac{d^2x}{dt^2}+mgl\sin x=0, \qquad (25)$$

where m is the mass of the pendulum, l the length, g the acceleration of gravity and x the angle of deflection from the vertical position.

For small angles of deflection, sin x may, with a sufficient degree of accuracy, be replaced by x. In this case, eq.(25) may be reduced to the form

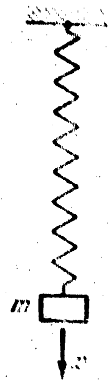

Fig.2

$$\frac{d^2x}{dt^2}+\frac{g}{l}x=0. \qquad (26)$$

Equation (26) is an equation of harmonic oscillations which will be isochronous, i.e., their period $T = 2\pi\sqrt{\frac{l}{g}}$ will not depend on the initial velocity nor on the deflection.

However, for large deflections of the pendulum, eq.(26) is inexact. In the case where the deflections do not exceed angles of the order of one radian, sin x in eq.(25) may be replaced, with sufficient accuracy, by the first two terms of the expansion into series:

$$\sin x = x - \frac{x^3}{6} + \ldots \qquad (27)$$

Equation (25) will then take the form

$$\frac{d^2x}{dt^2}+\frac{g}{l}\left(x-\frac{x^3}{6}+\ldots\right)=0. \qquad (28)$$

Obviously, as will be shown in detail below, the oscillations in this case will no longer be isochronous, and their frequency will depend on their amplitude of oscillation

$$\omega = \omega(a).$$

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Let us also consider the oscillations of a certain weight of mass m, suspended on a spring (Fig. 2). Let $f(x)$ be the force produced by a spring stretched to the

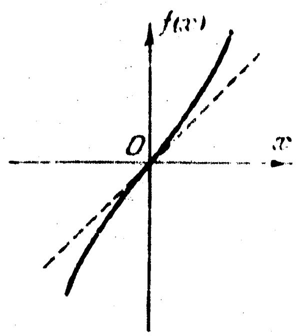

Fig. 3

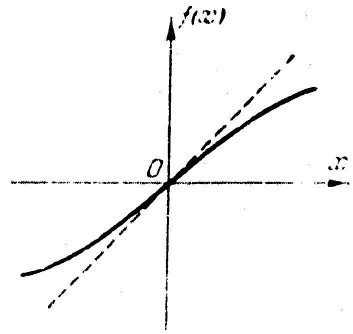

Fig. 4

length x with respect to its state of rest. In this case, the rigidity of the spring for the displacement x may be defined as $f'(x)$. Again disregarding the forc of friction, we have the equation of motion

$$m\frac{d^2x}{dt^2}+f(x)=0. \qquad (29)$$

If the rigidity of the spring increases with increasing displacement, then, a it is customary to say, the spring has a hard characteristic of nonlinear restori force (Fig. 3). If the rigidity decreases with increasin displacement, the spring has a soft characteristic (Fig Obviously the oscillations of the weight on the spring, scribed by eq. (29), will likewise not be isochronous, i view of the fact that the period of oscillation will d crease with increasing rigidity, while the effective r dity in this case will increase with the amplitude.

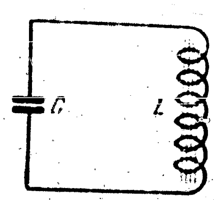

Fig. 5

Analogously, in the case of a soft characteristic, the period will decre For example, in eq. (28), the restoring force $f(x) = mgl\,(x - \frac{x^3}{6})$ has a s characteristic, and, consequently, the period of oscillation will increase wi

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creasing amplitude.

We now present an example of a nonlinear electric oscillatory system.

For this purpose, let us consider the oscillatory circuit consisting of the inductance L, an iron core, and the capacitance C (Fig. 5).

Let Φ be the magnetic flux flowing across the coil. Then the equation for the circuit under consideration may be written in the form

$$\frac{d^2\Phi}{dt^2} + \frac{i}{C} = 0, \qquad (30)$$

where i is the current strength. To determine the relationship between the current strength and the flux induced in the coil, when an iron core is present, let us use the following generally adopted formula:

$$i = a\Phi - b\Phi^3, \qquad (31)$$

where $a > 0$, $b > 0$.

Then, by substituting eq.(31) in eq.(30), we obtain the following nonlinear equation

$$\frac{d^2\Phi}{dt^2} + \frac{1}{C}(a\Phi - b\Phi^3) = 0, \qquad (32)$$

for which the restoring force likewise has a soft characteristic.

In the above oscillatory systems the force of friction, resulting in the damp-ing of the natural oscillations, has not been taken into account.

It is well known that the laws of mechanical friction, speaking generally, ha[ve] been studied only slightly.

In practice, the following laws of friction are ordinarily used:

1) The force of friction is proportional to the velocity (for oscillations i[n] air at low velocities);

2) The force of friction is proportional to the square of the velocity (for [os]cillations in a liquid medium or in air, but at high velocities);

3) The force of friction is constant in value, does not depend on the velo[city] and acts in a direction opposite the velocity; this is known as the Coulomb fri[ction]

4) The internal friction depends on the losses in the material under osci[llation]

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tions (in a spring, in the filament of a pendulum, etc.). In this case, the force of friction is usually expressed in the form of more complex relations with the displacement or velocity.

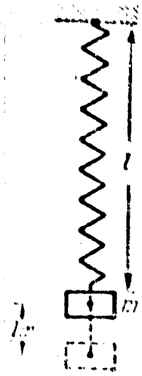

Fig.6

For example, on considering a pendulum freely swinging in air, and assuming that the force of friction is proportional to the velocity, we obtain the equation of oscillations in the form

$$\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + \frac{g}{l}\sin x = 0, \tag{33}$$

where λ is the factor of proportionality, usually called the damping factor.

If a body suspended on a spring, with a restoring force $f(x)$, is submerged in a liquid, then, assuming that the force of friction is proportional to the square of the velocity, we obtain the following equation:

$$m\frac{d^2x}{dt^2} \pm \alpha\left(\frac{dx}{dt}\right)^2 + f(x) = 0; \tag{34}$$

where α is the damping factor, and the plus sign should be taken for $\frac{dx}{dt} > 0$ and the minus sign for $\frac{dx}{dt} < 0$, since the force of friction is always directed opposite to the velocity of the body.

Let us assume further that, this same body suspended on the spring, is not subjected to the square-law friction but to Coulomb friction.

Then the equation of motion will be

$$m\frac{d^2x}{dt^2} + A\,\mathrm{sign}\left(\frac{dx}{dt}\right) + f(x) = 0, \tag{35}$$

where A denotes the absolute value of the force of friction, and

$$\mathrm{sign}\left(\frac{dx}{dt}\right) = \begin{cases} 1, & \frac{dx}{dt} > 0, \\ -1, & \frac{dx}{dt} < 0. \end{cases} \quad (3$$

As an example, illustrating the estimation of the forces of internal friction, us consider the vertical oscillations of a certain mass m suspended from a ro

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length l and cross section F, with a modulus of elasticity E (Fig.6)*.

Let us assume that the mass m can execute only vertical oscillations, while the mass of the rod, which in our case plays the role of the spring, is small by comparison with the suspended mass m. Then the system may be considered as a system with one degree of freedom. In setting up the differential equation of motion of the oscillating mass m, we will take into consideration the losses of energy of oscillation due to internal diffusion in the material of the rod, but we will neglect the mass of the rod itself.

Let x be the relative elongation of the rod. We then obtain an equation in the form

$$ml\frac{d^2x}{dt^2} + EFx + EF\overset{\leftrightarrows}{\Phi}(x) = 0, \tag{37}$$

or

$$\frac{d^2x}{dt^2} + \omega^2[x + \overset{\leftrightarrows}{\Phi}(x)] = 0, \tag{38}$$

where $\omega^2 = \frac{EF}{ml}$ is the natural frequency of the linear system, $\omega^2\overset{\leftrightarrows}{\Phi}(x)$ a function allowing for the damping in the material of the rod, its value being different for the ascending and descending motions. In the equation, this is indicat by the two opposite arrows.

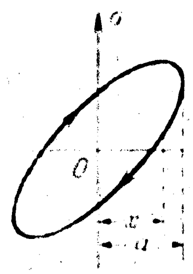

Fig.7

Thus the process of oscillation of the system under consideration, in which energy is being lost dissipation in the material of the rod, is in fac pressed by two differential equations.

We will assume that the deviation of the relation between the stress σ and deformation x in the material of the rod differs little from the linear Hooke

---

* This example has been considered in detail by G.S.Pisarenko (Bibl.34). In study, he developed a general method permitting the influence of internal tion to be studied in systems with either a finite or an infinite number o grees of freedom.

In this respect, eq.(38) will be close to linear.

We now present an explicit expression for the function $\omega^2\overset{\leftrightarrows}{\Phi}(x)$. Let us assume that, for vertical oscillations of the mass m, suspended on an elastic rod, the hysteresis loop constructed in the coordinates: x = relative elongation and σ = normal stress, will be symmetrical (Fig.7).

In this case, the true modulus of elasticity will be variable, and according to N.N.Davidenkov's hypothesis, the expression for the function $\omega^2\overset{\leftrightarrows}{\Phi}(x)$ will have the following form:

$$\left.\begin{aligned} \omega^2\overrightarrow{\Phi}(x) &= -\frac{\nu\omega^2}{n}\left[(a+x)^n - 2^{n-1}a^n\right], \\ \omega^2\overleftarrow{\Phi}(x) &= \frac{\nu\omega^2}{n}\left[(a-x)^n - 2^{n-1}a^n\right], \end{aligned}\right\} \qquad (39)$$

where ν and n are constants determined experimentally, while a is the amplitude of oscillations.

In the above-considered oscillatory cases, the dissipative forces (forces of friction) have sometimes not been taken into account; however, in reality the dissipative forces will always, to some extent, influence an oscillatory system, as a result of which the oscillations are damped in time.

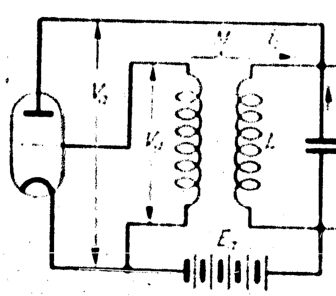

Fig.8

Undamped oscillations can, in practice, exist in a case where there is a certain source of energy in the system which is able to compensate the expenditure of energy created by the presence of dissipative forces.

Such a source of energy may be a periodic force acting on the oscillatory system. For example, if, an ordinary linear vibrator is subjected to an external periodic force consisting of a single harmonic, then we obtain the following equation of motion:

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$$m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx = E\sin\nu t, \tag{40}$$

according to which undamped oscillations will exist in the oscillatory system. In a given case, the losses due to friction caused by the presence of the dissipative term $\lambda\frac{dx}{dt}$, will be compensated because of the externally generated energy and characterized by the periodic term $E\sin\nu t$.

The source of energy in itself may have no definite periodicity, while its action on the oscillatory system plays a role similar to that of negative friction, which compensates the ordinary positive friction introduced by the dissipative forces.

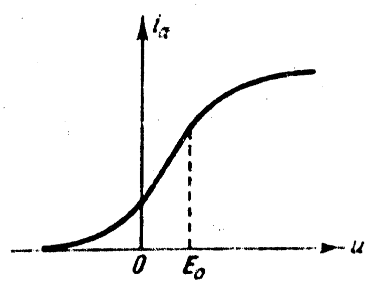

Fig.9

Oscillations of this type, which differ substantially from the above case of the presence of a periodic source of energy, are called self-sustained oscillations.

In self-sustained oscillatory systems, under certain conditions, the equilibrium position loses stability, and motion occurs, bringing the system into a state of stationary periodic oscillation (i.e., oscillations having a constant amplitude and phase).

For the realization of a stationary periodic state, the system must consist o three parts: an oscillatory system; a certain source of energy controlling the oscillatory system, whose action on the system compensates the losses due to frictio makes the position of equilibrium unstable, and causes the oscillations to incre and a certain limiter, which brings these increasing oscillations into a statio state.

The first two parts of the system may be linear, but the oscillation limit always nonlinear; therefore, any self-sustained oscillatory system is describe nonlinear differential equation.

Self-sustained oscillatory systems are widely encountered, and are very

portant in physics and engineering.

To get a clear idea on the character of the excitation of oscillations in a self-sustained oscillatory system, let us consider the oscillations of a system with one degree of freedom.

If the oscillations are of small amplitude, we can use the linear differential equation

$$m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx = 0. \tag{41}$$

The general solution of this equation, as is well known, is

$$x = ae^{-\delta t}\cos(\omega t + \varphi), \tag{42}$$

where a and φ are integration constants,

$$\delta = \frac{\lambda}{2m}, \quad \omega^2 = \frac{k}{m} - \left(\frac{\lambda}{2m}\right)^2. \tag{43}$$

Therefore, if $\lambda > 0$, the amplitude of the small oscillations $ae^{-\delta t}$ will obviously be damped by an exponential law. If, on the other hand, $\lambda < 0$, the small oscillations will continue, and their amplitude will increase by an exponential law.

In view of the fact that the amplitude of oscillation cannot increase without limit, it is natural to postulate that, beginning at some instant, the coefficient of damping must change its sign and become positive.

This property of an oscillatory system may be reflected in the differential equation of the oscillations by replacing the constant coefficient λ by a variable coefficient, for example, of the following form:

$$\lambda = -A + B\left(\frac{dx}{dt}\right)^2, \tag{44}$$

where $A > 0$, $B > 0$.

As a result, we obtain an equation of the form

$$m\frac{d^2x}{dt^2} + \left\{-A + B\left(\frac{dx}{dt}\right)^2\right\}\frac{dx}{dt} + kx = 0,$$

from which it follows that the damping is negative for small absolute values and positive for large absolute values. Thus, oscillations of small amplitu

continue, and those of large amplitude will be damped.

This indicates that there exist undamped self-sustained oscillations toward which oscillations with both small and large amplitudes tend.

Equation (45) is called the Rayleigh equation and is of major importance in the theory of self-sustained oscillations.

Equation (45), by substitution of the variables

$$\left.\begin{aligned} \tau &= t\sqrt{\frac{k}{m}}, \\ y &= \sqrt{\frac{3Bk}{Am}}\int x\,d\tau \end{aligned}\right\} \tag{46}$$

may be reduced to the following form:

$$\frac{d^2y}{d\tau^2} - \varepsilon(1 - y^2)\frac{dy}{d\tau} + y = 0, \tag{47}$$

where $\frac{A}{\sqrt{km}} = \varepsilon$.

The equation of self-sustained oscillations, represented in this form, is called the van der Pol equation.

We now present concrete examples of self-sustained oscillatory systems. Le consider the electronic oscillator schematically shown in Fig.8.

Let i_L, i_C, and i_R be, respectively, the currents of the inductance L, the capacitance C and the resistance R of an oscillatory circuit.

Let E_a be the constant voltage in the plate circuit; V_a be the total volt on the plate of a vacuum electron tube; V_g the grid voltage; i_a the plate curr M the coefficient of mutual inductance between the grid circuit and the oscil circuit.

According to the diagram in Fig.8, neglecting the grid current, we have

$$L\frac{di_L}{dt} = \frac{1}{C}\int^t i_C\,dt = Ri_R = E_a - V_a,$$

$$M\frac{di_L}{dt} = V_g,$$

$$i_a = i_L + i_R + i_C, \tag{50}$$

and, consequently, we can write the differential equation

$$LC\frac{d^2 i_L}{dt^2} + \frac{L}{R}\frac{di_L}{dt} + i_L = i_a. \tag{51}$$

As is well known, the plate current i_a is a definite function of the control voltage $u = V_g + DV_a$, i.e.

$$i_a = f(u) = f(V_g + DV_a), \tag{52}$$

where D is a constant called the grid through of the electron tube. The numerical value of D is usually small with respect to unity.

On substituting eq.(52) in eq.(51) and taking account of eq.(48) and eq.(49), we get

$$LC\frac{d^2 i_L}{dt^2} + \frac{L}{R}\frac{di_L}{dt} + i_L = f\left[DE_a + (M - DL)\frac{di_L}{dt}\right]. \tag{53}$$

Let us now consider the following quantities

$$E_0 = DE_a,$$

$$V = (M - LD)\frac{di_L}{dt}.$$

Obviously V is the alternating component of the control voltage u, excited b the oscillations of the current in the oscillatory circuit, while E_0 is the cons component, excited by the source of direct current.

Then, for an unknown V, eq.(53) yields an equation of the following form:

$$LC\frac{d^2V}{dt^2} + V + \left\{\frac{L}{R} - (M - LD)f'(E_0 + V)\right\}\frac{dV}{dt} = 0.$$

On substituting the variables by the formula

$$t = \tau\sqrt{LC},$$

we obtain the following equation for the electronic oscillator in dimensio form:

$$\frac{d^2V}{d\tau^2} + V + \frac{1}{\sqrt{LC}}\left\{\frac{L}{R} - (M - LD)f'(E_0 + V)\right\}\frac{dV}{d\tau} =$$

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and, consequently, we can write the differential equation

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On substituting the variables by the formula

$$t = \tau\sqrt{LC},$$

we obtain the following equation for the electronic oscillator in dimensio form:

$$\frac{d^2V}{d\tau^2} + V + \frac{1}{\sqrt{LC}}\left\{\frac{L}{R} - (M - LD)f'(E_0 + V)\right\}\frac{dV}{d\tau} =$$

Under certain assumptions, eq.(55) may be reduced to the van der Pol equation. Now let us assume that the constant component of the control voltage E_0 is se-

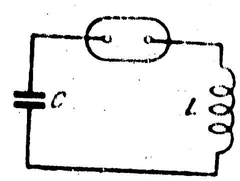

Fig. 10

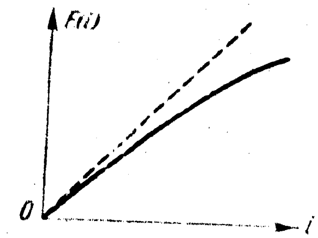

Fig. 11

lected in such a way that it forms the abscissa of the point of inflection of the tube characteristic:

$$i_a = f(u),$$

i.e., such that

$$f''(E_0) = 0$$

Then, for a certain interval of values of V, we may put, in first approximation:

$$f(E_0 + V) = f(E_0) + Vf'(E_0) + \frac{V^3}{6} f'''(E_0). \qquad (56)$$

Let us assume that

$$(M - LD) f'''(E_0) < 0,$$
$$(M - LD) f'(E_0) - \frac{L}{R} > 0.$$

Then eq.(55) is transformed into the van der Pol equation:

$$\frac{d^2x}{dt^2} - \varepsilon (1 - x^2) \frac{dx}{dt} + x = 0, \qquad (57)$$

in which the following notation is introduced:

$$\frac{(M - LD) f'(E_0) - \frac{L}{R}}{\sqrt{LC}} = \varepsilon, \quad \alpha V = x,$$

$$\alpha^2 = \frac{-(M - LD) f'''(E_0)}{2\left\{(M - LD) f'(E_0) - \frac{L}{R}\right\}}.$$

We now present still another example of a self-oscillatory system enc
in electrical engineering.

Consider the electric oscillatory circuit (Fig.10) consisting of the capacitance C, the inductance L, and a certain element with a nonlinear voltage-current characteristic: $e = F(i)$; (Fig.11).

Then the oscillations in the circuit will be described by the equation

$$LC\frac{d^2i}{dt^2} + i + CF'(i)\frac{di}{dt} = 0, \qquad (58)$$

or, if we assume that the characteristic is approximated by the fifth-degree polynomial

$$F(i) = A + Bi + Di^2 + Ei^3 + Fi^4 + Gi^5, \qquad (59)$$

by the following equation:

$$LC\frac{d^2i}{dt^2} + i + C(B + 2Di + 3Ei^2 + 4Fi^3 + 5Gi^4)\frac{di}{dt} = 0, \qquad (60)$$

for which, depending on the character of the polynomial of eq.(59), a solution corresponding to the stationary oscillatory state may exist.

Let us consider the Froude pendulum as a mechanical self-sustained oscillatory system.

The construction of this pendulum is schematically shown in Fig.12: The shaft O

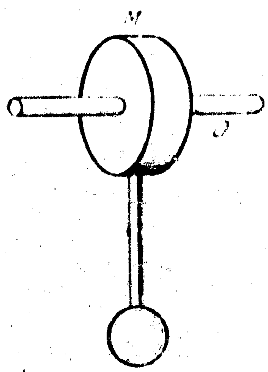

Fig. 12

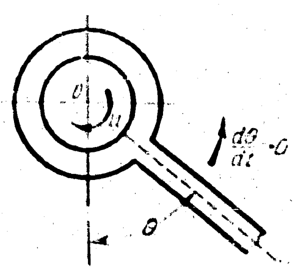

Fig. 13

rotating at uniform speed, carries the sleeve M rigidly connected with the [...] the friction of the sleeve while turning on the shaft is known. Let θ be [...]

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of deflection of the pendulum axis from the vertical (Fig. 13), u the angular velocity of rotation of the shaft, J the moment of inertia of the pendulum, $\lambda \frac{d\theta}{dt}$ the moment of the forces of friction of the pendulum against the air, which is proportional to the angular velocity of oscillation of the pendulum $\frac{d\theta}{dt}$.

In addition to the force of friction between the pendulum and the air, the moment of friction between the shaft and the sleeve must also be taken into account in setting up the equation.

Let us assume for simplification that this moment $\mathfrak{M}$ is a certain function F of the relative angular velocity v (of the pendulum with respect to the shaft)

$$\mathfrak{M} = F(v). \tag{61}$$

In the case under consideration, we obviously have

$$v = u + \frac{d\theta}{dt},$$

and, therefore, the equation of oscillations of the Froude pendulum (idealized) may be expressed in the form

$$J\frac{d^2\theta}{dt^2} + \lambda\frac{d\theta}{dt} + F\left(u + \frac{d\theta}{dt}\right) + mga\sin\theta = 0, \tag{62}$$

where m denotes the mass of the pendulum; a the distance from the center of grav of the pendulum to the center of the shaft axis and g the acceleration of gravi

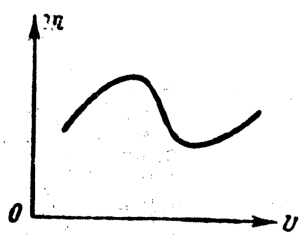

Fig. 14

The characteristic of friction be sleeve and shaft, represented by eq.(6 has, generally speaking, a descending (Fig. 14) on which

$$F'(v) < 0.$$

Let us select the velocity u in such that u forms the abscissa of the p inflection of the descending branch:

$$F''(u) = 0,$$

while assuming that

$$F'(u) + \lambda < 0,$$

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$$F'''(u) > 0.$$

For values of $\frac{d\theta}{dt}$ lying within certain limits, we may put

$$F\left(u + \frac{d\theta}{dt}\right) = F(u) + F'(u)\frac{d\theta}{dt} + \frac{1}{6}F'''(u)\left(\frac{d\theta}{dt}\right)^3. \quad (63)$$

Then, on considering the case of small oscillations, and using $\sin\theta \approx \theta$, we reduce eq.(62) to the Rayleigh equation or to the van der Pol equation.

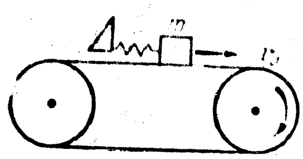

Fig.15

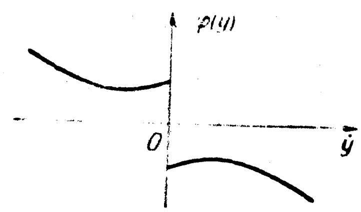

Fig.16

Below, we are given another example of a mechanical self-sustained oscillatory system:

Let us assume that a certain body of mass m is located on the rough surface of a belt, stretched between two pulleys and moving at constant velocity v_o (Fig.15). The body is attached to a fixed point by a spring whose elasticity varies accord to a linear law.

It is well known that, for certain values of v_o, the body will not be in a state of rest, but will perform discontinuous oscillations. This is due to the fact that the force of dry friction between the body and the belt is not a c quantity but varies with $\frac{dy}{dt}$, being the velocity with which the body slides the belt (Fig.16). So long as the body is at rest with respect to the belt $= 0$), the force of friction $\varphi\left(\frac{dy}{dt}\right)$ increases, remaining at each instant eq the force of attraction of the spring. When this force reaches a certain equal to the critical value of the force of friction, the body begins to respect to the belt; at first, the force of friction will decrease with

$\left|\frac{dy}{dt}\right|$ but will decrease as soon as $\left|\frac{dy}{dt}\right|$ reaches a sufficiently high value.

Let us take as the origin of coordinates at a point at which the spring is in its undeformed state. Then the sliding velocity of the body may be written in the form

$$\frac{dy}{dt} = \frac{dx}{dt} - v_0,$$

while the equation of motion of the body will have the following form:

$$m\frac{d^2x}{dt^2} + \varphi\left(\frac{dx}{dt} - v_0\right) + kx = 0. \qquad (64)$$

If, before the beginning of computation, a position of the body is taken at which it is in equilibrium under the simultaneous action of the elastic force and the force of friction, i.e., the position for which

$$\varphi(-v_0) + kx = 0, \qquad (65)$$

then, for the new coordinate z:

$$z = x + \frac{1}{k}\varphi(-v_0),$$

the equation of motion will have the form

$$m\frac{d^2z}{dt^2} + F\left(\frac{dz}{dt}\right) + kz = 0, \qquad (66)$$

where

$$F\left(\frac{dz}{dt}\right) = \varphi\left(\frac{dz}{dt} - v_0\right) - \varphi(-v_0).$$

The above oscillatory systems are close-to-linear.

In the differential equations obtained, the nonlinear terms, being small, ma[y] be considered as a weak disturbance proportional to some small parameter.

Thus, in the general case, for such oscillating systems, it is convenient t[o] write the differential equation in the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}\right), \qquad (6[7])$$

where the right side contains the small nonlinear terms ($\varepsilon \ll 1$) characterizi[ng]

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linear friction, nonlinear additions to the restoring elastic force, nonlinear terms resulting in the existence of a self-sustained oscillatory state, etc.

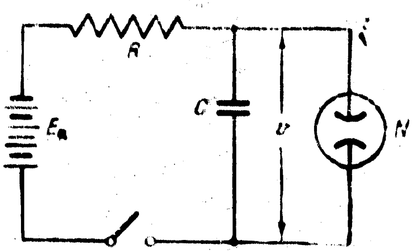

Fig. 17

The introduction of the small parameter ε is a very convenient means of mathematically representing the fact that, during the course of a period of time of the order of $\frac{2\pi}{\omega}$ (the period of oscillation) the form of the oscillations is close to sinusoidal.

If we speak of the general properti[es] of such nonlinear oscillations, it must [be] emphasized that, in contrast to linear oscillations, their frequencies and amplit[udes] may be variable. Even if we neglect the higher harmonics and put approximately

$$x = a\cos\theta, \qquad (68)$$

we must remember in this case that a and θ, speaking generally, are variables.

In nonlinear systems, the amplitude of oscillation may vary according to th[e] intake or loss of energy in the system. In weakly nonlinear systems this varia[tion] is naturally very small, so that, in practice, it will show up only after a ti[me] that is much greater than the time of a single cycle of oscillation.

On considering the expression for the instantaneous power produced by the forces represented in the equation by small disturbing nonlinear terms, we see this expression, being a periodic function of the phase angle θ, likewise dep[ends on] the amplitude a.

Since the phase angle rotates at a frequency ω, while its amplitude var[ies] rather slowly, it follows that when the mean is taken over a time interval [in] which the amplitude does not appreciably deviate from its initial value, wh[ile the] phase already has been able to make a rather large number of revolutions, [terms] containing cos nθ and sin nθ become very small, since the expressions

$$\frac{1}{T}\int_0^T \cos n\theta\, dt = \frac{1}{n\omega T}\sin n\theta\Big|_0^T,$$

$$\frac{1}{T}\int_0^T \sin n\vartheta\, dt = -\frac{1}{n\omega T}\cos n\vartheta \Big|_0^T \qquad (69)$$

contain T in the denominator.

Thus the power developed by these forces in the time interval T will depend only on the amplitude of oscillation.

It is therefore natural to consider the approximate equation, determining the course of the variation of amplitude, in the following form

$$\frac{da}{dt} = F(a). \qquad (70)$$

We will later demonstrate that the successive application of the asymptotic method actually leads to an equation of precisely this type.

It is expedient to represent the function F(a) in the form $-\delta(a)a$, by introducing the effective decrement $\delta(a)$, here emphasizing the analogy with linear systems in which δ is a constant.

It is thus natural to put

$$\left.\begin{aligned} \frac{da}{dt} &= -\delta(a)\,a, \\ \frac{d\vartheta}{dt} &= \omega(a), \end{aligned}\right\} \qquad (71)$$

where $\omega(a)$ is the "effective frequency of oscillation"; and $\delta(a)$ is the "effective damping", determined by the presence of an energy source or energy sink in the system.

In systems with friction, at any values of a,

$$\delta(a) > 0,$$

while, in self-sustained oscillatory systems, for certain values of the amplitude of oscillation

$$\delta(a) = 0.$$

Let us discuss another type of oscillatory systems which are widespread in [illegible]ture but differ substantially from the above self-sustained oscillatory system[illegible] slight nonlinearity, in that their parameter ε is a large parameter; in parti[illegible] let us consider the case $\varepsilon \to \infty$.

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The oscillations described by equations of this type are called relaxation oscillations. This term reflects the existence of two different and characteristic stages into which the oscillatory process is divided in the relaxation oscillatory

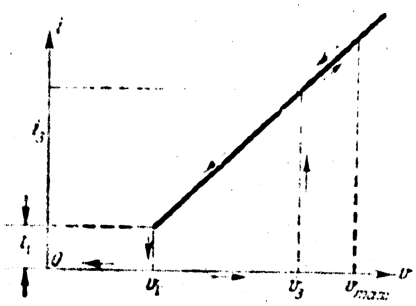

Fig. 18

system, namely the slow accumulation of energy of the system, and the subsequent discharge of that energy which latter occurs almost instantly after a certain critical potential threshold for this accumulation of energy has been reached.

Oscillatory systems of the relaxation type (ε being sufficiently great), just like the above quasi-linear systems, may be covered by rather general methods of study.

As an example, let us consider the relaxation oscillations taking place under such conditions in a circuit with a neon tube, arranged according to the circuit dia-gram in Fig.17. In this diagram, the neon tube N is connected to the DC voltage source E_a across the resistance R and is shunted by the capacitance C. In conside-ing this diagram, we will neglect the forces of inertia in the oscillatory system which leads us ultimately to differential equations of the first order instead of the second.

The nonlinearity of the oscillatory system presented in the diagram of Fig. is due to the neon tube, for which the relation between the voltage v and the cur-rent i does not obey the linear Ohm's law.

On idealization, the process of variation of the current i, as a functio[n] variation in the voltage v, proceeds as follows: When v deviates from the ...

toward increasing levels, the tube will not burn at first and carry no current. When a certain voltage v_{ig}, known as the ignition voltage, is reached, the tube lights, its resistance drops sharply, and the current jumps to a value of i_1; a further increase of v results in a linear increase of the current. If now, after the voltage has reached a certain value $v_{max} > v_{ig}$, we begin to reduce it continuously, the current i will decrease by a linear law so long as v does not reach what is called the extinguishing voltage v_{ex}. At this instant the tube goes dead and the current i is interrupted, becoming equal to zero. In the commonly adopted idealization, this relationship is graphically represented by the characteristic of Fig.18, which consists of linear segments and a segment with hysteresis loop.

For this reason we may put $i = 0$ for the extinguished tube and $i = i_{ex} + \frac{v - v_{ex}}{R_N}$ for the burning tube.

To set up the differential equation, we note that, since the current through the capacitor is equal to $C\frac{dv}{dt}$, the current in the resistance R will be $C\frac{dv}{dt} + i$, and, consequently, the voltage drop across R will be represented by the equation

$$R\left(C\frac{dv}{dt} + i\right) = E_a - v,$$

whence we find

$$\frac{dv}{dt} = \frac{E_a - v - Ri}{CR}. \qquad (72)$$

When the circuit is connected to the voltage E_a, the process of charging the capacitor begins.

Since, at the initial instant, the voltage across the tube terminals is equal to zero, it follows that, for a certain period of time, it will be less than v_{ig}. During this time the neon tube will not light, and, accordingly, eq.(72) will yiel

$$\frac{dv}{dt} = \frac{E_a - v}{RC}. \qquad (73)$$

If $E_a > v_{ig}$, then, according to eq.(73), v will reach the value v_{ig} after a cer time, and the tube will light; after this, at a sufficiently high value of R, t process of discharge of the capacitor across the neon tube begins. For this p we have

$$\frac{dv}{dt} = \frac{E_a - v - R\left[i_r + \frac{v - v_r}{R_N}\right]}{RC}. \qquad (74)$$

Let R be so great that

$$E_a - v - Ri_r < 0.$$

Then v, according to eq.(74), monotonously declines to the value $v = v_{ex}$, after which the tube extinguishes and the process of charging the capacitor resumes, followed again by discharge across the tube, and so on.

Thus, periodic relaxation oscillations take place in the circuit under consideration, causing v to vary between v_{ex} and v_{ig}.

The equation of oscillation may obviously be written in the form

$$\frac{dv}{dt} = \Phi(v), \qquad (75)$$

where $\Phi(v)$, over the interval $v_{ex} \leq v \leq v_{ig}$, according to eq.(73) and (74), has the two values:

$$\left.\begin{aligned} \Phi(v) &= \frac{E_a - v}{RC} \quad \text{(for increasing v)} \\ \Phi(v) &= \frac{E_a - v - R \cdot i_r - \frac{R}{R_N}(v - v_r)}{RC} \\ &\quad \text{(for decreasing v)} \end{aligned}\right\} \qquad (76)$$

3. In the above examples of oscillatory systems, the disturbing forces did n depend explicitly on the time. These oscillatory systems were isolated from external influences, as a result of which all the forces acting on the system depended only on the dynamic state of the system itself.

Below, we will present typical examples of nonlinear oscillatory systems u the influence of external periodic forces, and will analyze, from the physical of view, the characteristic phenomena that can arise in nonlinear systems in case.

Let us begin with the consideration of the simplest example of all, a oscillator subjected to the weak harmonic exciting force $\varepsilon E \cos \nu t$.

The equation of motion in this case is written in the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}\right) + \varepsilon E \cos \nu t. \qquad (77)$$

In view of the smallness of the external force and the smallness of the nonlinear term, the oscillations after one cycle will have a form close to harmonic:

$$\left.\begin{aligned} x &\approx a \cos(\omega t + \psi), \\ \frac{dx}{dt} &\approx -a\omega \sin(\omega t + \psi). \end{aligned}\right\} \qquad (78)$$

The amplitudes and phases will undergo substantial modifications in this case only after a time covering a large number of cycles.

Let us consider the mean power

$$\frac{1}{T}\int_{t_0}^{t_0+T} \varepsilon E \cos \nu t \cdot \frac{dx}{dt}\, dt, \qquad (79)$$

introduced into the system by an external force over the time T, during which it still does not succeed in noticeably modifying the form of the oscillations. Since the external force is considered small, this time T, during which the oscillation is approximately "natural", may be taken sufficiently large with respect to the cycle of oscillation.

On substituting eq.(78) in eq.(79) we obtain the mean power in the form

$$A = \frac{a\omega\varepsilon E}{2T}\left[\frac{\cos[(\nu + \omega)t + \psi]}{\nu + \omega} - \frac{\cos[(\nu - \omega)t - \psi]}{\nu - \omega}\right]_{t_0}^{t_0+T}. \qquad (80)$$

For sufficiently large values of T, this expression is practically different from zero only when the external frequency is sufficiently close to the natural fre-quency.

Thus a small disturbing force, acting for a prolonged period of time, may ex[illegible] a perceptible influence on the oscillator under consideration only in the case of resonance $\nu \approx \omega$. Let us agree to call such resonance the "principal" or ordinar[illegible] resonance.

In our discussion we started out from the purely harmonic form of natural [illegible]cillations.

However, because of the presence of a nonlinear term in the natural osci[illegible]

higher harmonic are unavoidably generated (even in the absence of an external exciting force).

On substituting x in eq.(79) by the expression for the natural oscillation containing higher harmonics of frequencies $n\omega$ and by repeating the above reasoning, we prove that the mean power under consideration, for a sufficiently great T, differs from zero not only at $\omega \approx \nu$, but also at $n\omega \approx \nu$, where $n = 2, 3,$ etc. These auxiliary resonances, which can be detected only in the next approximation, in allowing for the presence of overtones in the natural oscillations, will be called frequency-division resonance or demultiplication resonance.

In still higher approximations, more complex resonances of the fractional type

$$n\omega \approx m\nu$$

may be detected. These are due to the fact that, at a sufficiently high approximation, the expression for $\frac{dx}{dt}$ must take account not only of the overtones of the natural frequency, but also of the combination tones with frequencies of the type $n\omega \pm r\nu$.

In the case under discussion of a nonlinear oscillator with a small applied harmonic force, we detected, in first approximation, only the principal resonance $\omega \approx \nu$. In certain weakly nonlinear systems, already in first approximation, resonances of a different type appear, for example, submultiple resonances.

For example, if we consider an oscillatory system whose rigidity varies periodically, we obtain the well-known Mathieu equation

$$\frac{d^2x}{dt^2} + \omega^2(1 + h\cos\nu t)\,x = 0. \qquad (81)$$

Assuming the coefficient of frequency modulation $\omega^2 h$ to be small, and consid[ering] the term

$$\omega^2 h\cos\nu t \cdot x \qquad (82)$$

to be an external excitation, let us calculate the expression for the mean po[wer].

Here, already in first approximation, we have

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$$
\begin{aligned}
A &= \frac{1}{T}\int_{t_0}^{t_0+T} \omega^2 h \cos \nu t \cos(\omega t + \psi)\, a\omega \sin(\omega t + \psi)\, dt = \\
&= \frac{\omega^3 h}{4T}\left[\frac{\cos[(\nu + 2\omega)t + 2\psi]}{\nu + 2\omega} - \frac{\cos[(\nu - 2\omega)t - 2\psi]}{\nu - 2\omega}\right]_{t_0}^{t_0+T}
\end{aligned}
\qquad (83)
$$

and, consequently, the work performed by the disturbing force of eq.(82), taken in its mean value over a long interval of time, does not in practice vanish in the case when $\nu \approx 2\omega$, i.e., $\omega \approx \frac{\nu}{2}$.

Thus, in the oscillatory system described by eq.(81), we are able, already in first approximation, to detect the resonance of frequency halving.

Let us also consider the influence of the external harmonic excitation on the self-sustained oscillatory system. For this purpose let us take the generalized van der Pol equation in the form

$$\frac{d^2x}{dt^2} - \varepsilon(1 - x^2)\frac{dx}{dt} + x = E \sin \nu t. \qquad (84)$$

Assuming that ν does not take values close to unity, let us perform the substitution of variables by the expression

$$x = y + U \sin \nu t, \qquad (85)$$

where $U = \frac{E}{1 - \nu^2}$.

Then we have

$$\frac{d^2y}{dt^2} + y = \varepsilon[1 - (y + U \sin \nu t)^2]\left[\frac{dy}{dt} + U\nu \cos \nu t\right]. \qquad (86)$$

Let us analyze what resonances can be detected in first approximation for the given equations:

On calculating the mean power produced by a small disturbing force

$$\varepsilon[1 - (y + U \sin \nu t)^2]\left[\frac{dy}{dt} + U\nu \cos \nu t\right] \qquad (87)$$

in a state of purely harmonic oscillation

$$y = a \cos(\omega t + \psi),$$

$$\frac{dy}{dt} = -a\omega \sin(\omega t + \psi),$$

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it is not obvious that it does not vanish for $2\omega \approx \nu$, and likewise for $2\nu \approx \omega$ whereas the resonance $2\omega \approx \nu$ originates in the component of the disturbing force of eq. (87) at an amplitude proportional to U, while the resonance $2\nu \approx \omega$ originates in the component whose amplitude is proportional to U^2.

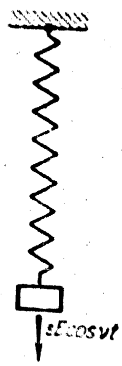

Fig. 19

Thus, if E is a small quantity, the resonance $2\omega \approx \nu$ can be detected in second approximation but the resonance $2\nu \approx \omega$, only in third approximation.

From this preliminary analysis it is clear that, in cases of resonance, only weak periodic perturbations can exert a substantial influence on the course of the oscillatory process during a rather long interval of time.

In this case, in distinction to perturbations not explicitly dependent on time, of the type $\varepsilon f\left(x, \frac{dx}{dt}\right)$, the phase ratios will play a substantial role.

Now, if we take the typical equation (80) and use $\omega = \nu$ for simplicity, we find:

$$A \approx -\frac{a\omega\varepsilon E}{2}\sin\psi. \tag{88}$$

Thus, at $\sin\psi < 0$, energy is received by the system, and at $\sin\psi > 0$, energy is taken from the system. Already this simple example indicates the importance of phase relations in the case of resonance.

For this reason, the approximate equations determining the course of the oscillations must include not only the amplitude, but also the phase of the oscillation, so that, instead of a single expression (70), we have now a system of two conjugate equations of the type

$$\left.\begin{aligned} \frac{da}{dt} &= F(a, \psi), \\ \frac{d\psi}{dt} &= \Phi(a, \psi). \end{aligned}\right\} \tag{89}$$

The question of the factual determination of the right-hand sides of these

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equations with respect to the first and higher approximations will be considered in Chapter III. In that Chapter we will describe computational asymptotic methods confirming the accuracy of the preliminary arguments of a qualitative character here presented, and properly developing these arguments.

In the above examples we had to do with oscillatory systems described by equations of the type

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(\nu t,\ x,\ \frac{dx}{dt}\right), \tag{90}$$

where $f(\nu t, x, \frac{dx}{dt})$ is a periodic function with the period of 2π with respect to νt.

In case of a nonlinear oscillator we have

$$\varepsilon f\left(\nu t,\ x,\ \frac{dx}{dt}\right) = \varepsilon f\left(x,\ \frac{dx}{dt}\right) + \varepsilon E \cos \nu t.$$

In the case of eq.(81), the Mathieu equation, we have

$$\varepsilon f\left(\nu t,\ x,\ \frac{dx}{dt}\right) = -\omega^2 h \cos \nu t \cdot x.$$

It must be emphasized that a large number of practically important oscillatory systems can be expressed by an equation of this type.

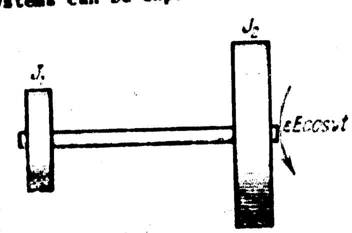

Fig. 20

As an example of an oscillatory system described by eq.(77), we may take the problem of the oscillation of a weight of unit mass, suspended from a spring with a nonlinear elasticity characteristic, in the presence of friction, with the spring being under the influ[ence] of the disturbing force $\varepsilon E \cos \nu t$ (Fig. 19).

The problem of the torsional oscillations of a shaft, consisting of two ma[sses] joined by a nonlinear elastic tie, one of which masses is subjected to the si[nu]soidal torsional moment $F(t) = \varepsilon E \cos \nu t$ (Fig. 20), leads to an equation of t[he]

$$J_1 J_2 \frac{d^2\theta}{dt^2} + (J_1 + J_2)\, c(\theta) + f\left(\frac{d\theta}{dt}\right) = \varepsilon E \cos \nu t,$$

which is likewise of the type of eq.(77). Here J_1 and J_2 denote moments [of inertia] of the masses $\theta = \theta_1 - \theta_2$ is the angle of torsion and $f(\frac{d\theta}{dt})$ is a functi[on of the] velocity, taking account of the influence of friction.

The problem of the oscillations of a pendulum under the action of an external periodic force is likewise reduced to an equation of the type of eq.(77), etc.

An equation of the Mathieu type (81) expresses, for example, the problem of the transverse oscillations of a hinged rod subjected to the action of the longitudinal pulsating force $F(t) = \varepsilon E \cos \nu t$.

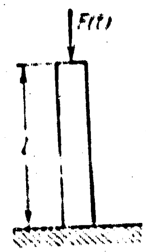

Fig. 21

In this case we obtain the equation

$$\frac{d^2x}{dt^2} + \omega^2\left[1 - \frac{l^2\varepsilon E_1}{EJ\pi}\cos \nu t\right]x = 0, \qquad (92)$$

in which the following symbols are introduced: EJ = rigidity of rod; l = length of rod (Fig. 21).

An equation of a more general type than the Mathieu equation is found in the consideration of a nonlinear oscillator in which certain parameters, for example the natural frequency of the corresponding linear system, vary periodically.

We then have an equation of the form

$$\frac{d^2x}{dt^2} + \omega^2(1 + h \sin \nu t)\,x = \varepsilon f\left(x, \frac{dx}{dt}\right). \qquad (93)$$

As an example, let us consider the oscillations in the circuit consisting of the ordinary inductance L, a coil with the characteristic (magnetic flux to current) of the form

$$\Phi = \Phi(i),$$

of low ohmic resistance R, and the variable capacitance

$$C = C_0(1 + \rho \sin \nu t)$$

Then the total flux of magnetic induction in the circuit will be

$$Li + \Phi(i),$$

In this way, we have the following equation:

$$\frac{d\,[Li + \Phi(i)]}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = 0,$$

or, introducing, as the unknown charge on the capacitor plates:

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$$q = \int_0^t i\,dt,$$

we obtain:

$$\left\{L + \Phi'\left(\frac{dq}{dt}\right)\right\}\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \qquad (95)$$
$$+ \frac{q}{C_0(1 + \rho\sin\nu t)} = 0.$$

Assuming that the inductance L is sufficiently great relative to the "nonlinear inductance"

$$L \gg \Phi'(i)$$

and, in addition, that the coefficient ρ, characterizing the "depth of variation" of the capacitance, is sufficiently small, eq.(95) may, with accuracy to terms of the second order of smallness, be brought into the form:

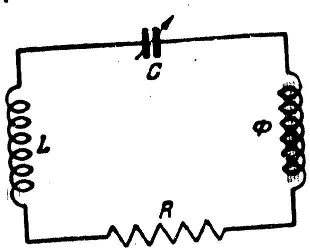

Fig. 22

$$\frac{d^2q}{dt^2} + \omega^2(1 - \rho\sin\nu t)\,q =$$
$$= -\frac{R}{L}\left(1 - \frac{\Phi'\left(\frac{dq}{dt}\right)}{L}\right)\frac{dq}{dt} + \omega^2\frac{\Phi\left(\frac{dq}{dt}\right)}{L}q, \qquad (96)$$

where $\omega = \frac{1}{\sqrt{LC}}$.

As an example of a self-sustained oscillatory system, under the influence [of] an external sinusoidal excitation, let us consider the regenerative receiver schematically in Fig.23.

In this case, we get the equation

$$LC\frac{d^2i}{dt^2} + RC\frac{di}{dt} + i = i_a,$$

while

$$i_a = f(V_a)$$

is the characteristic of the tube.

Since

$$V_a = M_r\frac{di}{dt} + E_0\sin\nu t,$$

where $M_r = M - DL$, then by substituting

$$i_1 = i - \frac{E_0}{M_r \nu} \cos \nu t \tag{100}$$

eq.(97) can be brought into the form

$$CL \frac{d^2 i_1}{dt^2} + RC \frac{di_1}{dt} + i_1 = f\left(M_r \frac{di_1}{dt}\right) + E_1 \cos(\nu t + \varphi), \tag{101}$$

where

$$E_1 = \frac{E_0 C \sqrt{(LC\nu^2 - 1)^2 + R^2 C^2 \nu^2}}{M_r \nu}.$$

Equation (101) belongs to the type of eq.(93), and can, by the substitution

$$i_1 = I + \frac{E_1}{\omega^2 - \nu^2} \cos(\nu t + \varphi), \tag{102}$$

where $\omega^2 = \frac{1}{LC}$ be brought into the form of eq.(86).

Up to now we have been considering the influence of external periodic forces on oscillatory systems close to linear. We may, however, also consider the question of

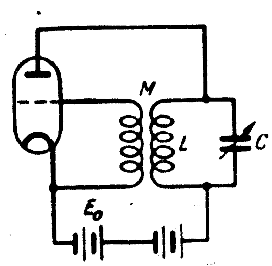

Fig.23

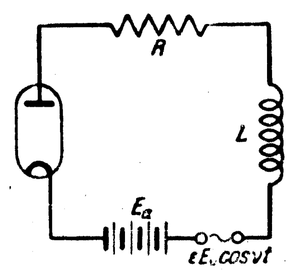

Fig.24

the influence of small periodic disturbing forces on the relaxation oscillatory sys-tems, described, for example, by an equation of the type

$$\frac{dx}{dt} = \Phi(x) + \varepsilon E \cos \nu t, \qquad (10$$

where, as in the case of eq.(75) for free relaxation oscillations, $\Phi(x)$ repre- a two-value function defined over the interval (a,b).

In particular, we obtain eq.(103), if in the previously considered diagram of the generator of relaxation oscillations, we connect the alternating harmonic voltage $\varepsilon E_\nu \cos \nu t$ in series with the constant voltage E_a.

Thus, for the diagram given in Fig.24, we obviously have:

$$\frac{di}{dt} = \Phi(i) + \frac{\varepsilon E_\nu \cos \nu t}{L}. \qquad (104)$$

In view of the fact that the overtones, up to a very high order, play a substantial role in free relaxation oscillations, an analysis of the expression

$$\frac{1}{T}\int_0^T \varepsilon E_\nu \cos \nu t \cdot \frac{dx}{dt}\,dt,$$

will show the system contains resonances of the frequency-division type

$$\nu = \frac{\omega}{n},$$

where n may be a rather large number.

For this reason the frequency-division resonances may be observed in large numbers and most conveniently, in relaxation systems.

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CHAPTER I

NATURAL OSCILLATIONS IN QUASI-LINEAR SYSTEMS

Section 1. Construction of Asymptotic Solutions

Let us discuss now the method of constructing asymptotic approximations, firs of all for the case of the oscillations defined by a differential equation of the type

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}\right), \qquad (1.1)$$

where ε is a small positive parameter.

We may approach the correct formulation of this method by starting from the physical ideas of the character of the oscillatory process under consideration.

Thus, in the absence of disturbance, i.e., at $\varepsilon = 0$, the oscillations will viously be purely harmonic

$$x = a \cos \psi$$

with constant amplitude and uniformly rotating phase angle:

$$\frac{da}{dt} = 0, \quad \frac{d\psi}{dt} = \omega \quad (\psi = \omega t + \theta)$$

(the amplitude a and the phase θ of the oscillation will be quantities cons time, and will depend on the initial conditions).

The presence of a nonlinear disturbance ($\varepsilon \neq 0$) will lead to the appe overtones in the solution of eq.(1.1), will cause the instantaneous freque to depend on the amplitude, and, finally, may produce a systematic increa crease in the amplitude of oscillation, according to the inflow or absorp energy by the disturbing forces.

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All these effects of the perturbation will obviously vanish in the limiting case ($\varepsilon = 0$).

Taking all this into consideration, let us find the general solution of eq.(1.1) in the form of the expansion*:

$$x = a\cos\psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \varepsilon^3 u_3(a,\psi) + \cdots, \quad (1.2)$$

where $u_1(a,\psi)$, $u_2(a,\psi)$,...are periodic functions of the angle ψ with the period 2π, while the quantities a, ψ, as functions of time, are defined by the differential equations

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \cdots, \\ \frac{d\psi}{dt} &= \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \cdots \end{aligned}\right\} \quad (1.3)$$

Thus we now have the task of selecting the appropriate expression for the functions $u_1(a,\psi)$, $u_2(a,\psi)$,..., $A_1(a)$, $B_1(a)$, $A_2(a)$, $B_2(a)$,...in such a way that equation (1.2) in which the functions of time defined by eq.(1.3) are substituted for a and ψ, will represent the solution of the original eq.(1.1).

As soon as this solution is solved and explicit expressions are found for the coefficients of the expansions in the right-hand sides of eq.(1.2) and eq.(1.3), the question of the integration of eq.(1.1) will finally consist of the simpler question of the integration of eq.(1.3) with separable variables, thus allowing study by the aid of the well-known elementary methods.

As shown later, there are no fundamental difficulties in determining the coefficients of these expansions, but in view of the rapidly increasing complexity of the formulas, only the first two or three terms can be properly defined in actual work.

Concentrating in our expansions on these terms, i.e., assuming that

$$x = a\cos\psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \cdots + \varepsilon^m u_m(a,\psi), \quad (m = 1, 2, \ldots) \quad (1.$$

* This formulation of the method of expansion by a small parameter was first in the book by Krylov and N.N.Bogolyubov "Introduction to Nonlinear Mech (Bibl.21).

where

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots + \varepsilon^m A_m(a), \\ \frac{d\psi}{dt} &= \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots + \varepsilon^m B_m(a), \end{aligned}\right\} \quad (1.5)$$

we will be able to obtain approximations of the first, second, and so on low order; therefore, the practical application of the method is determined not by the properties of convergence of the sums of eqs.(1.4) and (1.5), as $m \to \infty$, but by their asymptotic properties for a given fixed m, as $\varepsilon \to 0$. All that is required is that, for a small value of ε, eq.(1.4) yield a sufficiently exact representation of the solution of eq.(1.1) over a sufficiently long time interval. For this reason, the problem of convergence, as $m \to \infty$, will be disregarded here and the expansions of eq.(1.2) and (1.3) will be considered as formal expansions necessary for the construction of the asymptotic approximations (1.4).

In other words, let us pose the problem more cautiously by formulating it as the problem of determining the functions

$$u_1(a, \psi),\ u_2(a, \psi),\ \ldots,\ A_1(a),\ A_2(a),\ \ldots,\ B_1(a),\ B_2(a),\ \ldots, \quad (1.6)$$

such that eq.(1.4), in which the time functions a, ψ are determined by "equations of the m^{th} approximation" (1.5), satisfies eq.(1.1) with an accuracy to terms of the ε^{m+1} order of smallness.

We remark that it is precisely in this case of eq.(1.1) that the convergence of the expansions (1.2) and (1.3) could be established under very general conditions imposed on the function $f(x, \frac{dx}{dt})$. Since, however, we later will have to do with cases in which analogous expansions are known to diverge, we are not linking our method of constructing asymptotic approximations to the proofs of convergence; therefore, everywhere and without specific reservations, we will assign the above stated formal meaning to the series arranged by powers of a small parameter.

Let us say a few more words on evaluating the error. From the fact that the approximate solution obtained satisfies eq.(1.1) with an error of the ε^{m+1} order, the usual majorant method can be used for proving that the deviation of the

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mate solution from the corresponding exact solution (where the initial conditions are in agreement) is limited to a quantity of the order of $\varepsilon^{m+1}t$, and that, consequently, this deviation will remain small for any assigned values of εt, no matter how large, provided only that ε itself is sufficiently small. Here we have the fundamental difference from the approximate formulas of eq.(6), considered in the introduction, which were applicable only at small values of εt, i.e., only over a time interval during which the amplitude of oscillation does not perceptibly depart from its initial value.

We would like to mention that the question of a strict justification of the asymptotic methods constitutes a special and purely mathematical problem which is of significance for the theory of differential equations with a small parameter.

For this reason, we consider it expedient to relegate their discussion to Chapter V which is a mathematical supplement to the main text of the book. Here, however, we are concentrating our attention on the problem of the actual construction of approximate solutions and their application to the study of specific examples. On a number of examples, for which the exact solutions are known, we will illustrate the effectiveness of the method and the accuracy of the approximate formulas.

Before taking up the problem of constructing approximate solutions, we would like to mention that the problem of determining eq.(1.6) contains a certain element of the arbitrary.

Let us assume that certain expressions for these functions have been found: Taking the arbitrary functions

$$\alpha_1(a),\ \alpha_2(a),\ \ldots,\ \beta_1(a),\ \beta_2(a),\ \ldots,$$

and performing, in eq.(1.2) and eq.(1.3) the substitution of the variables:

$$a = b + \varepsilon\alpha_1(b) + \varepsilon^2\alpha_2(b) + \ldots,$$
$$\psi = \varphi + \varepsilon\beta_1(b) + \varepsilon^2\beta_2(b) + \ldots,$$

we obtain

$$x = b\cos\varphi +$$
$$+ \varepsilon\,[\alpha_1(b)\cos\varphi - b\beta_1(b)\sin\varphi + u_1(b,\ \varphi)] + \varepsilon^2 \ldots,$$

$$
\left.\begin{aligned}
\frac{db}{dt} &= \varepsilon A_1(b) + \\
&+ \varepsilon^2\left[\frac{dA_1(b)}{db}\gamma_1(b) - \frac{d\gamma_1(b)}{db}A_1(b) + A_2(b)\right] + \varepsilon^3 \ldots, \\
\frac{d\varphi}{dt} &= \omega + \varepsilon B_1(b) + \\
&+ \varepsilon^2\left[\frac{dB_1(b)}{db}\gamma_1(b) - \frac{d\gamma_1(b)}{db}A_1(b) + B_2(b)\right] + \varepsilon^3 \ldots
\end{aligned}\right\} \quad (1.7)
$$

As indicated by eq.(1.7), we again arrive at equations of the type of eqs.(1.2) and (1.3), but with different expressions for the coefficients (1.6). In order to have these coefficients determined uniquely, we must impose additional conditions on them, which may be done, generally speaking, with a certain degree of the arbitrary. As such additional conditions, let us take the condition that the first harmonic is absent from the expressions $u_1(a,\psi)$, $u_2(a,\psi)$,... In other words, these periodic functions of the phase angle will be so expressed as to satisfy the equations

$$
\left.\begin{aligned}
\int_0^{2\pi} u_1(a,\psi)\cos\psi\, d\psi = 0, \quad \int_0^{2\pi} u_2(a,\psi)\cos\psi\, d\psi = 0, \ \ldots \\
\int_0^{2\pi} u_1(a,\psi)\sin\psi\, d\psi = 0, \quad \int_0^{2\pi} u_2(a,\psi)\sin\psi\, d\psi = 0, \ \ldots
\end{aligned}\right\} \quad (1.8)
$$

From the physical point of view, the adoption of these conditions corresponds to the selection of the full amplitude of the first fundamental harmonic of the oscillation as the quantity a.

After these preliminary remarks, let us discuss the problem involved, namely to find the appropriate expressions for eq.(1.6), allowing for the additional conditions (1.8).

Differentiating the right-hand side of eq.(1.2), we get:

$$
\begin{aligned}
x &= a\cos\psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \ldots, \\
\frac{dx}{dt} &= \frac{da}{dt}\left\{\cos\psi + \varepsilon\frac{\partial u_1}{\partial a} + \varepsilon^2\frac{\partial u_2}{\partial a} + \ldots\right\} + \\
&\qquad + \frac{d\psi}{dt}\left\{-a\sin\psi + \varepsilon\frac{\partial u_1}{\partial \psi} + \varepsilon^2\frac{\partial u_2}{\partial \psi} + \ldots\right\}, \\
\frac{d^2x}{dt^2} &= \frac{d^2a}{dt^2}\left\{\cos\psi + \varepsilon\frac{\partial u_1}{\partial a} + \varepsilon^2\frac{\partial u_2}{\partial a} + \ldots\right\} +
\end{aligned}
$$

$$
\left.
\begin{aligned}
&+\frac{d^2\psi}{dt^2}\left\{-a\sin\psi+\varepsilon\frac{\partial u_1}{\partial\psi}+\varepsilon^2\frac{\partial u_2}{\partial\psi}+\dots\right\}+\\
&+\left(\frac{da}{dt}\right)^2\left\{\varepsilon\frac{\partial^2 u_1}{\partial a^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial a^2}+\dots\right\}+\\
&+2\frac{da}{dt}\frac{d\psi}{dt}\left\{-\sin\psi+\varepsilon\frac{\partial^2 u_1}{\partial a\,\partial\psi}+\varepsilon^2\frac{\partial^2 u_2}{\partial a\,\partial\psi}+\dots\right\}+\\
&+\left(\frac{d\psi}{dt}\right)^2\left\{-a\cos\psi+\varepsilon\frac{\partial^2 u_1}{\partial\psi^2}+\varepsilon^2\frac{\partial^2 u_2}{\partial\psi^2}+\dots\right\}.
\end{aligned}
\right\}\quad (1.9)
$$

Taking eq.(1. 3) into consideration, we may find the following quantities

$$
\left.
\begin{aligned}
\frac{d^2a}{dt^2}&=\left(\varepsilon\frac{dA_1}{da}+\varepsilon^2\frac{dA_2}{da}+\dots\right)(\varepsilon A_1+\varepsilon^2A_2+\dots)=\\
&=\varepsilon^2A_1\frac{dA_1}{da}+\varepsilon^3\dots,\\
\frac{d^2\psi}{dt^2}&=\left(\varepsilon\frac{dB_1}{da}+\varepsilon^2\frac{dB_2}{da}+\dots\right)(\varepsilon A_1+\varepsilon^2A_2+\dots)=\\
&=\varepsilon^2A_1\frac{dB_1}{da}+\varepsilon^3\dots,\\
\left(\frac{da}{dt}\right)^2&=(\varepsilon A_1+\varepsilon^2A_2+\dots)^2=\varepsilon^2A_1^2+\varepsilon^3\dots,\\
\frac{da}{dt}\frac{d\psi}{dt}&=(\varepsilon A_1+\varepsilon^2A_2+\dots)(\omega+\varepsilon B_1+\varepsilon^2B_2+\dots)=\\
&=\varepsilon A_1\omega+\varepsilon^2(A_2\omega+A_1B_1)+\varepsilon^3\dots,\\
\left(\frac{d\psi}{dt}\right)^2&=(\omega+\varepsilon B_1+\varepsilon^2B_2+\dots)^2=\\
&=\omega^2+\varepsilon 2\omega B_1+\varepsilon^2(B_1^2+2\omega B_2)+\varepsilon^3\dots
\end{aligned}
\right\}\quad (1.10)
$$

On substituting eqs.(1.3) and (1.10) in eq.(1.9), and arranging the result in powers of the parameter ε, we find:

$$
\begin{aligned}
\frac{dx}{dt}&=-a\omega\sin\psi+\varepsilon\left\{A_1\cos\psi-aB_1\sin\psi+\omega\frac{\partial u_1}{\partial\psi}\right\}+\\
&+\varepsilon^2\left\{A_2\cos\psi-aB_2\sin\psi+A_1\frac{\partial u_1}{\partial a}+B_1\frac{\partial u_1}{\partial\psi}+\right.\\
&\left.+\omega\frac{\partial u_2}{\partial\psi}\right\}+\varepsilon^3\dots,\\
\frac{d^2x}{dt^2}&=-a\omega^2\cos\psi+\varepsilon\left\{-2\omega A_1\sin\psi-2\omega aB_1\cos\psi+\right.
\end{aligned}
$$

(1.

$$+\omega^2 \frac{\partial^2 u_1}{\partial\psi^2}\Big\} + \varepsilon^2 \Big\{\Big(A_1 \frac{dA_1}{da} - aB_1^2 - 2\omega a B_2\Big)\cos\psi - \\ - \Big(2\omega A_2 + 2A_1B_1 + A_1\frac{dB_1}{da}a\Big)\sin\psi + \\ + 2\omega A_1 \frac{\partial^2 u_1}{\partial a\,\partial\psi} + 2\omega B_1 \frac{\partial^2 u_1}{\partial\psi^2} + \omega^2\frac{\partial^2 u_2}{\partial\psi^2}\Big\} + \varepsilon^3 \dots,$$

whence it follows that the left-hand side of eq.(1.1) may be represented in the form:

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon\Big\{-2\omega A_1\sin\psi - 2\omega a B_1\cos\psi + \\ + \omega^2\frac{\partial^2 u_1}{\partial\psi^2} + \omega^2 u_1\Big\} + \varepsilon^2\Big\{\Big(A_1\frac{dA_1}{da} - aB_1^2 - 2\omega a B_2\Big)\cos\psi - \\ - \Big(2\omega A_2 + 2A_1B_1 + A_1\frac{dB_1}{da}a\Big)\sin\psi + 2\omega A_1\frac{\partial^2 u_1}{\partial a\,\partial\psi} + \\ + 2\omega B_1\frac{\partial^2 u_1}{\partial\psi^2} + \omega^2\frac{\partial^2 u_2}{\partial\psi^2} + \omega^2 u_2\Big\} + \varepsilon^3 \dots \tag{1.12}$$

The right-hand side of eq.(1.1), taking eqs.(1.2) and (1.11) into consideration, may be written as follows:

$$\varepsilon f\Big(x, \frac{dx}{dt}\Big) = \varepsilon f(a\cos\psi, -a\omega\sin\psi) + \\ + \varepsilon^2\Big\{u_1 f'_x(a\cos\psi, -a\omega\sin\psi) + \Big(A_1\cos\psi - aB_1\sin\psi + \omega\frac{\partial u_1}{\partial\psi}\Big)\times \\ \times f'_{x'}(a\cos\psi, -a\omega\sin\psi)\Big\} + \varepsilon^3 \dots \tag{1.13}$$

To have this expression (1.2) satisfy the initial equation (1.1) with an accuracy to terms of the ε^{m+1} order of smallness, it is necessary to equate the coefficients of similar powers of ε in the right-hand sides of eqs.(1.12) and (1.13) to terms of the m^{th} order inclusive.

As a result, we obtain the following equations:

$$\omega^2\Big(\frac{\partial^2 u_1}{\partial\psi^2} + u_1\Big) = f_0(a, \psi) + 2\omega A_1\sin\psi + 2\omega a B_1\cos\psi,$$
$$\omega^2\Big(\frac{\partial^2 u_2}{\partial\psi^2} + u_2\Big) = f_1(a, \psi) + 2\omega A_2\sin\psi + 2\omega a B_2\cos\psi,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\omega^2\left(\frac{\partial^2 u_m}{\partial\psi^2}+u_m\right)=f_{m-1}(a,\psi)+2\omega A_m\sin\psi+2\omega aB_m\cos\psi,$$

where, for brevity, we introduced the following symbols:

$$\left.\begin{aligned}
&f_0(a,\psi)=f(a\cos\psi,\ -a\omega\sin\psi),\\
&f_1(a,\psi)=u_1f'_x(a\cos\psi,\ -a\omega\sin\psi)+\Big[A_1\cos\psi-\\
&\qquad -aB_1\sin\psi+\omega\frac{\partial u_1}{\partial\psi}\Big]f'_{x'}(a\cos\psi,-a\omega\sin\psi)+\\
&+\left(aB_1^2-A_1\frac{dA_1}{da}\right)\cos\psi+\left(2A_1B_1+A_1\frac{dB_1}{da}a\right)\sin\psi-\\
&\qquad\qquad -2\omega A_1\frac{\partial^2u_1}{\partial a\,\partial\psi}-2\omega B_1\frac{\partial^2u_1}{\partial\psi^2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\}\qquad(1.15)$$

It is obvious that $f_k(a,\psi)$ is a periodic function of the variable ψ with a period 2π depending on a; its explicit expression will be known as soon as the expressions $A_j(a)$, $B_j(a)$, $u_j(a,\psi)$ have been found up to the k^{th} number inclusive.

In order to determine $A_1(a)$, $B_1(a)$, $u_1(a,\psi)$ from the first equation of the system (1.14), let us consider the Fourier expansion for the functions $f_0(a,\psi)$ and $u_1(a,\psi)$:

$$\left.\begin{aligned}
f_0(a,\psi)&=g_0(a)+\sum_{n=1}^{\infty}\{g_n(a)\cos n\psi+h_n(a)\sin n\psi\},\\
u_1(a,\psi)&=v_0(a)+\sum_{n=1}^{\infty}\{v_n(a)\cos n\psi+w_n(a)\sin n\psi\}.
\end{aligned}\right\}\qquad(1.16)$$

On substituting the right-hand sides of eq. (1.16) in the first equation of the system (1.14), we obtain the expression

$$\omega^2v_0(a)+\sum_{n=1}^{\infty}\omega^2(1-n^2)\{v_n(a)\cos n\psi+w_n(a)\sin n\psi\}=$$
$$=g_0(a)+\{g_1(a)+2\omega aB_1\}\cos\psi+\{h_1(a)+2\omega A_1\}\sin\psi+$$
$$+\sum_{n=2}^{\infty}\{g_n(a)\cos n\psi+h_n(a)\sin n\psi\},$$

from which, by equating the coefficients of the same harmonics, we find

$$\left.\begin{array}{c} g_1(a)+2\omega a B_1=0, \quad h_1(a)+2\omega A_1=0, \\ v_0(a)=\frac{g_0(a)}{\omega^2}, \quad v_n(a)=\frac{g_n(a)}{\omega^2(1-n^2)}, \quad w_n(a)=\frac{h_n(a)}{\omega^2(1-n^2)}, \\ (n=2,\ 3,\ \ldots). \end{array}\right\} \quad (1.17)$$

Thus we have uniquely determined $A_1(a)$ and $B_1(a)$, as well as all the harmonic components of the function $u_1(a,\psi)$ except for the first ones $v_1(a)$ and $w_1(a)$. By virtue of the additional conditions (1.8), however, this function will not contain a first harmonic, so that

$$v_1(a)=0, \quad w_1(a)=0$$

and, consequently,

$$u_1(a,\ \psi)=\frac{g_0(a)}{\omega^2}+\frac{1}{\omega^2}\sum_{n=2}^{\infty}\frac{g_n(a)\cos n\psi+h_n(a)\sin n\psi}{1-n^2}. \quad (1.18)$$

By completely determining $u_1(a,\psi)$, $A_1(a)$ and $B_1(a)$, we obtain, in accordance with eq.(1.15), an explicit expression for $f_1(a,\psi)$. On expanding it into a Fourier series

$$f_1(a,\ \psi)=g_0^{(1)}(a)+\sum_{n=1}^{\infty}\{g_n^{(1)}(a)\cos n\psi+h_n^{(1)}(a)\sin n\psi\}$$

and making use of the second equation in the system (1.14) and of the condition (1.8), we find in an analogous way,

$$g_1^{(1)}(a)+2\omega a B_2=0, \quad h_1^{(1)}(a)+2\omega A_2=0 \quad (1.19)$$

and

$$u_2(a,\ \psi)=\frac{g_0^{(1)}(a)}{\omega^2}+\frac{1}{\omega^2}\sum_{n=2}^{\infty}\frac{g_n^{(1)}(a)\cos n\psi+h_n^{(1)}(a)\sin n\psi}{1-n^2}.$$

Thus we obtain a process for the successive and unique determination of the quantities (1.6) with which we are concerned.

This method allows the determination of

$$u_n(a,\ \psi),\quad A_n(a),\quad B_n(a) \quad (n=1,\ 2,\ 3,\ \ldots)$$

to any desired value of the index n, no matter how high, and thus the constru

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of approximate solutions satisfying eq.(1.1) with an accuracy to terms of any desired order of smallness with respect to ε.

As indicated by the process of determining the functions (1.6), the quantities $A_n(a)$, $B_n(a)$ ($n = 1, 2, 3, \ldots$) are uniquely determined by eq.(1.8), which expresses the absence of the first harmonic in the functions $u_n(a,\psi)$. As a result, we obtain for $A_n(a)$, $B_n(a)$, expressions of the type of eq.(1.17) or eq.(1.19), which assure the absence, from the right-hand sides of eq.(1.14), of terms having first harmonics, which in turn makes it possible to avoid the appearance of secular terms in the solution.

Let us consider the first approximation

$$x = a\cos\psi + \varepsilon u_1(a, \psi), \tag{1.20}$$

in which

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a), \\ \frac{d\psi}{dt} &= \omega + \varepsilon B_1(a). \end{aligned}\right\} \tag{1.21}$$

We note that, starting from eq.(1.21), we may write:

$$\Delta a = a(t) - a(0) \sim \varepsilon t \widetilde{A}_1,$$

$$\Delta(\psi - \omega t) = [\psi(t) - \omega t] - \psi(0) \sim \varepsilon t \widetilde{B}_1,$$

where $\widetilde{A}_1$ and $\widetilde{B}_1$ are certain mean values of $A_1(a)$ and $B_1(a)$ over the interval $(0,t)$. On considering the latter expressions, we see that the time t during which the quantities a and $\psi - \omega t$ can take finite increments must be of the order of $\frac{1}{\varepsilon}$.

On the other hand, the equations of the first approximation (1.21) are obtained after neglecting the terms of the order of smallness ε^2 in eq.(1.3); however such an error in the values of the first derivatives $\frac{da}{dt}$, $\frac{d\psi}{dt}$ over the time t will lead to an error of the order $\varepsilon^2 t$ in the values of the functions a and ψ themsel[ves]. Consequently, in this interval of time, during which a, $\psi - \omega t$ are able to be d[evi]ated considerably from their initial values, the errors in the values of the [ampli]tude and phase of the oscillations will be quantities of the order ε; therefo[re in] this interval, it is meaningless to retain the term $\varepsilon u_1(a,\psi)$ of the first or[der]

smallness in eq.(1.20), since the error of eq.(1.20) and the error of the simplified formula

$$x = a\cos\psi$$

will both be quantities of the first order of smallness.

Let us consider the case of stationary oscillations, i.e., of oscillati... ing place at constant amplitude and frequency. In this case, obviously,

$$\frac{da}{dt} = 0$$

or

$$\varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots = 0.$$

Because of the presence of the factor ε in front of $A_1(a)$, we see t... terms starting with the second order of smallness are rejected in eq.(1,... the value of the stationary amplitude of the first harmonic is determine... equation of first approximation

$$A_1(a) = 0,$$

then an error is committed which, speaking generally, no longer is of ... der but of the first order of smallness.

Taking all this into consideration, it will be natural, in the fu... as the first approximation the simplified expression

$$x = a\cos\psi,$$

in which a and ψ are defined by the equations

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a), \\ \frac{d\psi}{dt} &= \omega + \varepsilon B_1(a). \end{aligned}\right\}$$

By entirely analogous reasoning, we take, as the second approxim... pression

$$x = a\cos\psi + \varepsilon u_1(a, \psi),$$

in which the time functions a and ψ are defined by the expressions

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$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a), \\ \frac{d\psi}{dt} &= \omega + \varepsilon B_1(a). \end{aligned}\right\}$$

By entirely analogous reasoning, we take, as the second approxi pression

$$x = a \cos \psi + \varepsilon u_1(a, \psi),$$

in which the time functions a and ψ are defined by the expressions

$$\left.\begin{aligned}\frac{da}{dt} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a),\\ \frac{d\psi}{dt} &= \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a).\end{aligned}\right\} \tag{1.26}$$

Let us also derive here explicit formulas for $A_1(a)$, $A_2(a)$, $B_1(a)$, $B_2(a)$, and $u_1(a,\psi)$. From eqs. (1.15), (1.16), and (1.17), we have

$$\left.\begin{aligned}A_1(a) &= -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\sin\psi\, d\psi,\\ B_1(a) &= -\frac{1}{2\pi a\omega}\int_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\cos\psi\, d\psi.\end{aligned}\right\} \tag{1.27}$$

It follows further from eq. (1.18), that

$$u_1(a,\psi) = \frac{g_0(a)}{\omega^2} - \frac{1}{\omega^2}\sum_{n=2}^{\infty}\frac{g_n(a)\cos n\psi + h_n(a)\sin n\psi}{n^2-1}, \tag{1.28}$$

where $g_n(a)$ and $h_n(a)$ are found from the expressions

$$\left.\begin{aligned}g_n(a) &= \frac{1}{2\pi}\int_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\cos n\psi\, d\psi,\\ h_n(a) &= \frac{1}{2\pi}\int_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\sin n\psi\, d\psi.\end{aligned}\right\} \tag{1.29}$$

Finally, by virtue of eqs. (1.15) and (1.19), we may write

$$\begin{aligned}A_2(a) &= -\frac{1}{2\omega}\left\{2A_1B_1 + A_1\frac{dB_1}{da}a\right\} -\\ &\quad -\frac{1}{2\pi\omega}\int_0^{2\pi}\Big[u_1(a,\psi)f'_x(a\cos\psi,\ -a\omega\sin\psi) +\\ &\quad + \left(A_1\cos\psi - aB_1\sin\psi + \omega\frac{\partial u_1}{\partial\psi}\right)\times\\ &\quad \times f'_{x'}(a\cos\psi,\ -a\omega\sin\psi)\Big]\sin\psi\, d\psi,\\ B_2(a) &= -\frac{1}{2\omega}\left\{B_1^2 - \frac{A_1}{a}\frac{dA_1}{da}\right\} -\\ &\quad -\frac{1}{2\pi\omega a}\int_0^{2\pi}\Big[u_1(a,\psi)f'_x(a\cos\psi,\ -a\omega\sin\psi) +\end{aligned}$$

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$$
\left.\begin{aligned}
&+\left(A_1\cos\psi - aB_1\sin\psi + \omega\frac{\partial u_1}{\partial\psi}\right)\times \\
&\qquad\times f'_{x'}(a\cos\psi,\ -a\omega\sin\psi)\Big]\cos\psi\, d\psi,
\end{aligned}\right\}
$$

We note that the equation of the second approximation (1.26), where $A_2(a)$ and $B_2(a)$ are determined by eq.(1.30), are complicated, in view of the fact that they are written in the most general case. For concrete oscillatory systems, as will be shown below, they can be considerably simplified.

Let us give more detailed consideration to the first approximation:

In accordance with eqs.(1.23), (1.24), and (1.27), the first approximation for the solution of eq.(1.1) may be represented in the form

$$x = a\cos\psi, \tag{1.31}$$

where a and ψ are defined by the equations

$$
\left.\begin{aligned}
\frac{da}{dt} &= \varepsilon A_1(a), \\
\frac{d\psi}{dt} &= \omega + \varepsilon B_1(a) = \omega_1(a),
\end{aligned}\right\} \tag{1.32}
$$

while

$$\varepsilon A_1(a) = -\frac{\varepsilon}{2\pi\omega}\int\limits_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\sin\psi\, d\psi, \tag{1.33}$$

$$\omega_1(a) = \omega - \frac{\varepsilon}{2\pi a\omega}\int\limits_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\cos\psi\, d\psi. \tag{1.34}$$

We note that the resultant equations of first approximation coincide with the equations found by the van der Pol methods.

Let us derive still another formula to determine the instantaneous natural frequency $\omega_1(a)$.

By squaring the right-hand side of eq.(1.34), we obtain

$$
\begin{aligned}
\omega_1^2(a) = \omega^2 &- \frac{\varepsilon}{\pi a}\int\limits_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\cos\psi\, d\psi + \\
&+ \frac{\varepsilon^2}{4\pi^2 a^2\omega^2}\left[\int\limits_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\cos\psi\, d\psi\right]^2,
\end{aligned}
$$

whence, since we are making all calculations in first approximation and reject all terms of the second order of smallness with respect to ε, we have:

$$\omega_1^2(a) = \frac{1}{\pi a}\int_0^{2\pi} [\omega^2 a \cos\psi - \varepsilon f(a\cos\psi, -a\omega\sin\psi)]\cos\psi\, d\psi. \quad (1.35)$$

The expression for the square of the instantaneous natural frequency, eq.(1.35), may be simplified.

Let us introduce the function

$$F\left(x, \frac{dx}{dt}\right) = \omega^2 x - \varepsilon f\left(x, \frac{dx}{dt}\right),$$

combining the "quasi-elastic term" with the nonlinear term.

Then, obviously,

$$\omega_1^2(a) = \frac{1}{\pi a}\int_0^{2\pi} F(a\cos\psi, -a\omega\sin\psi)\cos\psi\, d\psi. \quad (1.36)$$

Analogously, bearing in mind that

$$\int_0^{2\pi} \omega^2 a \cos\psi \sin\psi\, d\psi = 0,$$

we have

$$\varepsilon A_1(a) = -\frac{1}{2\pi\omega}\int_0^{2\pi} F(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi. \quad (1.37)$$

In eqs.(1.36) and (1.37) so obtained, the functions $\varepsilon A_1(a)$ and $\omega_1(a)$ are repre- sented directly by the function $F(x, \frac{dx}{dt})$, which distinguishes them from eqs.(1.33) and (1.34), where only a nonlinear correction term is used. The quantity ω enter- ing into the last two formulas may obviously be interpreted as the approximate va- ue (0^{th} approximation) of the oscillation frequency of the system under conside- tion.

Let us derive still another method of obtaining an equation of the first proximation: First, we recall that, for $\varepsilon = 0$, eq.(1.1) allows the solution

$$x = a\cos\psi,$$

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$$\frac{dx}{dt} = -a\omega \sin\psi, \Big\}$$

where $\psi = \omega t + \theta$, while the amplitude a and the oscillation phase θ are constants.

It is obvious that eq.(1.38) may be retained even in the case of $\varepsilon \neq 0$, under the condition that we consider the quantities a and θ to be certain functions of time rather than constants.

Let eq.(1.38) be a certain substitution of variables and assume that a and θ (the amplitude and phase of the oscillation) are unknown functions of time; on determining these from eq.(1.38), we will find the required expression for the original unknown x. To set up the differential equations for a and θ, let us differentiate both sides of the first equation of the system (1.38). We get

$$\frac{dx}{dt} = \frac{da}{dt}\cos\psi - a\frac{d\theta}{dt}\sin\psi - a\omega\sin\psi, \qquad (1.39)$$

From this, taking into consideration the second relation of eq.(1.38), we obtain

$$\frac{da}{dt}\cos\psi - a\frac{d\theta}{dt}\sin\psi = 0. \qquad (1.40)$$

By differentiating both sides of the second equation of (1.38), we get

$$\frac{d^2x}{dt^2} = -\frac{da}{dt}\omega\sin\psi - a\omega\frac{d\theta}{dt}\cos\psi - a\omega^2\cos\psi. \qquad (1.41)$$

On substituting in eq.(1.1), the terms x, $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$ by their respective values obtained from eq.(1.38) and (1.41), we find

$$-\omega\frac{da}{dt}\sin\psi - a\omega\frac{d\theta}{dt}\cos\psi = \varepsilon f(a\cos\psi, -a\omega\sin\psi). \qquad (1.42)$$

By solving the system of two equations (1.40) and (1.42) with respect to the unknowns $\frac{da}{dt}$ and $\frac{d\theta}{dt}$, we obtain

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon}{\omega} f(a\cos\psi, -a\omega\sin\psi)\sin\psi, \\ \frac{d\theta}{dt} &= -\frac{\varepsilon}{a\omega} f(a\cos\psi, -a\omega\sin\psi)\cos\psi. \end{aligned}\right\} \qquad (1.43)$$

Thus, instead of the single second-order differential equation (1.1) with respect to the variable x, we now have the two first-order differential equations

with respect to the variable a and θ.

We observe that the right-hand sides of eq.(1.43) possess a period equal to $\frac{2\pi}{\omega}$, with respect to the independent variable t, and that in addition, $\frac{da}{dt}$ and $\frac{d\theta}{dt}$ are proportional to the small parameter ε, since a and θ will be slowly varying functions of time.

The differential equations, reduced to such a form, will be called equations of standard form.

It is obvious that the right-hand sides of eq.(1.43) may be represented in the form of the sums

$$\left.\begin{aligned} -\frac{\varepsilon}{\omega} f(a\cos\psi,\ -a\omega\sin\psi)\sin\psi &= \\ &= \varepsilon\sum_{\nu}[f_{\nu 1}^{(1)}(a)\cos\nu\psi + f_{\nu 2}^{(1)}(a)\sin\nu\psi], \\ -\frac{\varepsilon}{a\omega} f(a\cos\psi,\ -a\omega\sin\psi)\cos\psi &= \\ &= \varepsilon\sum_{\nu}[f_{\nu 1}^{(2)}(a)\cos\nu\psi + f_{\nu 2}^{(2)}(a)\sin\nu\psi]. \end{aligned}\right\} \quad (1.44)$$

According to the above, the form of the approximate solution of the system of equations may be determined from the following considerations: Since a and θ are slowly varying quantities, let us represent them as the superposition of the smooth-ly varying terms $\bar{a}$ and $\bar{\theta}$ and the sum of small vibrational terms. In first approxi-mation, we use

$$a = \bar{a}, \quad \theta = \bar{\theta}, \quad (\bar{\psi} = \omega t + \bar{\theta}). \quad (1.$$

Then,

$$\left.\begin{aligned} \frac{d\bar{a}}{dt} &= -\frac{\varepsilon}{\omega} f(\bar{a}\cos\bar{\psi},\ -\bar{a}\omega\sin\bar{\psi})\sin\bar{\psi} = \\ &= \varepsilon\sum_{\nu}[f_{\nu 1}^{(1)}(\bar{a})\cos\nu\bar{\psi} + f_{\nu 2}^{(1)}(\bar{a})\sin\nu\bar{\psi}], \\ \frac{d\bar{\theta}}{dt} &= -\frac{\varepsilon}{\bar{a}\omega} f(\bar{a}\cos\bar{\psi},\ -\bar{a}\omega\sin\bar{\psi})\cos\bar{\psi} = \\ &= \varepsilon\sum_{\nu}[f_{\nu 1}^{(2)}(\bar{a})\cos\nu\bar{\psi} + f_{\nu 2}^{(2)}(\bar{a})\sin\nu\bar{\psi}], \end{aligned}\right.$$

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or

$$\left.\begin{aligned} \frac{d\bar{a}}{dt} &= \varepsilon f_{01}^{(1)}(\bar{a}) + \text{small sinusoidal oscillatory terms} \\ \frac{d\bar{\theta}}{dt} &= \varepsilon f_{01}^{(2)}(\bar{a}) + \text{small sinusoidal oscillatory terms} \end{aligned}\right\} \qquad (1.47)$$

Considering that these sinusoidal oscillatory terms will produce only small vibrations of a and θ about their first approximations $\bar{a}$ and $\bar{\theta}$ and will not exert any influence on the systematic variation of a and θ, we arrive at the equations of first approximation in the form

$$\left.\begin{aligned} \frac{d\bar{a}}{dt} &= \varepsilon f_{01}^{(1)}(\bar{a}) = M_t\left\{-\frac{\varepsilon}{\omega} f(\bar{a}\cos\bar{\psi}, -\bar{a}\omega\sin\bar{\psi})\sin\bar{\psi}\right\}, \\ \frac{d\bar{\theta}}{dt} &= \varepsilon f_{01}^{(2)}(\bar{a}) = M_t\left\{-\frac{\varepsilon}{\bar{a}\omega} f(\bar{a}\cos\bar{\psi}, -\bar{a}\omega\sin\bar{\psi})\cos\bar{\psi}\right\}, \end{aligned}\right\} \qquad (1.48)$$

where M_t is the operator of the mean for the constants $\bar{a}$ and θ with respect to explicitly contained time.

It is obvious that the resultant eq. (1.48) for $\bar{a}$ and $\bar{\theta}$ coincides with the previously found equations of first approximation.

Indeed, by taking the mean and introducing the full phase of the oscillations ψ instead of θ, we obtain:

$$\frac{da}{dt} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi,$$

$$\frac{d\psi}{dt} = \omega - \frac{\varepsilon}{2\pi a\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\, d\psi.$$

To obtain the second approximation we must also consider the vibrational terms in the expressions for a and θ. By taking account, in eq. (1.47), of the terms $\varepsilon f_{\nu 1}^{(i)}(\bar{a})\cos\nu\bar{\psi}$, $\varepsilon f_{\nu 2}^{(i)}(\bar{a})\sin\nu\bar{\psi}$ $(i = 1, 2)$ since they produce in a and θ oscillations of the form

$$\frac{\varepsilon\sin\nu\bar{\psi}}{\nu} f_{\nu 1}^{(i)}(\bar{a}), \quad \frac{\varepsilon\cos\nu\bar{\psi}}{\nu} f_{\nu 2}^{(i)}(\bar{a}), \quad (i = 1, 2),$$

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we arrive at the following approximate expressions:

$$\left.\begin{aligned} a &= \bar{a} + \varepsilon \sum_{\nu \neq 0} \frac{1}{\nu} \left[f^{(1)}_{\nu 1}(\bar{a}) \sin \nu\bar{\psi} - f^{(1)}_{\nu 2}(\bar{a}) \cos \nu\bar{\psi} \right], \\ \theta &= \bar{\theta} + \varepsilon \sum_{\nu \neq 0} \frac{1}{\nu} \left[f^{(2)}_{\nu 1}(\bar{a}) \sin \nu\bar{\psi} - f^{(2)}_{\nu 2}(\bar{a}) \cos \nu\bar{\psi} \right], \end{aligned}\right\} \qquad (1.49)$$

which correspond to the refined first approximation. On substituting the values of eq.(1.49) in the right-hand side of eq.(1.38) we obtain eq.(1.25) with an accuracy to terms of the first order of smallness, inclusive.

To obtain the equation of the second approximation, determining a and θ with an accuracy to terms of the second order of smallness inclusive, the values of equation (1.49) must be substituted in the right-hand side of eq.(1.48) and the mean of the result taken with respect to explicitly contained time.

The above reasoning may be given a better substantiated form. For this purpose, eqs.(1.45) and (1.49), etc. must be considered as a substitution of variables in the original system of differential equations reduced to standard form. This question will be discussed in greater detail at the end of this book.

We note here that this method of taking the mean of differential equations, reduced to the standard form, greatly facilitates the application of the methods of nonlinear mechanics to the construction of approximate solutions of systems of nonlinear differential equations.

In this Section we described a method of constructing approximate solutions for equations of the type of eq.(1.1). There are no difficulties in extending this method to an equation of the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}, \varepsilon\right), \qquad (1.50)$$

where

$$\varepsilon f\left(x, \frac{dx}{dt}, \varepsilon\right) = \varepsilon f_1\left(x, \frac{dx}{dt}\right) + \varepsilon^2 f_2\left(x, \frac{dx}{dt}\right) + \dots$$

Here the right-hand side of the equation is dependent on ε in a more complex ma In view of the absence of special peculiarities, the latter equation will not b further discussed; instead, a more detailed consideration of various special c

met in practice will be given.

Section 2. Conservative Quasi-Linear Systems

As a special case of eq.(1.1), let us consider the free pseudoharmonic undamped oscillations of a certain mass m, i.e., the oscillations described by an equation of the form

$$m\frac{d^2x}{dt^2} + p(x) = 0, \qquad (2.1)$$

in which the relation

$$F_{н} = -p(x)$$

between the elastic force and the displacement is nonlinear.

Let us assume that this nonlinearity is sufficiently "weak", so that we may put

$$p(x) = kx + \varepsilon\Phi(x). \qquad (2.2)$$

Then eq.(2.1) will belong to the type under consideration, and

$$\omega^2 = \frac{k}{m}, \quad f\left(x, \frac{dx}{dt}\right) = -\frac{\Phi(x)}{m}, \qquad (2.3)$$

where ε is a small positive parameter.

For constructing the first approximation, let us consider the Fourier expansion for the function Φ (a cos ψ). Since this function is even, the sines will be absent from its expansion into a Fourier series:

$$\Phi(a\cos\psi) = \sum_{n=0}^{\infty} C_n(a)\cos n\psi. \qquad (2.4)$$

From this, eqs.(1.16) and (2.3) will yield

$$g_n(a) = -\frac{C_n(a)}{m}, \quad h_n(a) = 0,$$

whence

$$A_1(a) = 0, \quad B_1(a) = -\frac{1}{2\omega ma}C_1(a). \qquad (2.5)$$

Thus, taking account of eqs.(1.23) and (1.24), we have, in first approximation,

$$x_1 = a\cos\psi,$$

where a and ψ are determined by the equations

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$$\frac{da}{dt} = 0, \quad \frac{d\psi}{dt} = \omega + \frac{\varepsilon C_1(a)}{2\omega m a} = \omega_1(a) \tag{2.6}$$

[the subscripts of y, x, and $\omega(a)$ indicate the number of approximation].

It follows from the first equation of the system (2.6) that the amplitude of the oscillation does not depend on the time and preserves its initial value

$$a = a_0 = \text{const}.$$

In view of the constancy of a, we obtain, from the second equation of (2.6):

$$\psi = \omega_1(a)t + \theta,$$

where θ is a phase constant equal to the initial value of the phase ψ.

Thus in this case the oscillation studied in first approximation, will be harmonic. The nonlinear character of eq.(2.1) in first approximation has obviously only the effect of making the frequency of oscillation $\omega_1(a)$ depend on the amplitude. In other words, owing to the presence of the nonlinear term $\varepsilon\Phi(x)$ in equation (2.1), the oscillatory system loses its isochronism (isochronism is the term applied to the property of linear oscillatory systems of having a frequency of natural oscillation independent of the value of the amplitude), while, as follows from the expression for $\omega_1(a)$ in eq.(2.6), the loss of isochronism will be smaller, the smaller $\varepsilon\Phi(x)$ is by comparison with $\omega^2 x$.

Let us now discuss the construction of the second approximation: From eq.(1.18) we find

$$u_1(a, \psi) = \frac{1}{k}\sum_{\substack{n=0 \\ n\neq 1}}^{\infty} \frac{C_n(a)\cos n\psi}{n^2-1}. \tag{2.7}$$

On substituting eqs.(2.5) and (2.7) so found in eq.(1.30), we get

$$\left.\begin{aligned} &A_2(a) = 0, \\ &B_2(a) = -\frac{1}{2\omega}\left[\frac{C_1(a)}{2\omega m a}\right]^2 + \\ &\quad + \frac{1}{2\omega m\pi a}\sum_{\substack{n=0 \\ n\neq 1}}^{\infty} \frac{C_n(a)}{k(n^2-1)}\int_0^{2\pi} \Phi'(a\cos\psi)\cos\psi\cos n\psi\, d\psi. \end{aligned}\right\} \tag{2.8}$$

Since

$$C_0(a)=\frac{1}{2\pi}\int_0^{2\pi}\Phi(a\cos\psi)\,d\psi,$$

$$C_n(a)=\frac{1}{\pi}\int_0^{2\pi}\Phi(a\cos\psi)\cos n\psi\,d\psi,\quad (n\geq 1)$$

we obtain by differentiation:

$$\frac{1}{2\pi}\int_0^{2\pi}\Phi'(a\cos\psi)\cos\psi\,d\psi=\frac{dC_0(a)}{da},$$

$$\frac{1}{\pi}\int_0^{2\pi}\Phi'(a\cos\psi)\cos\psi\cos n\psi\,d\psi=\frac{dC_n(a)}{da}$$

Consequently, we may write

$$B_2(a)=-\frac{1}{2\omega}\left[\frac{C_1(a)}{2\omega ma}\right]^2+\frac{1}{2\omega mka}\left\{\sum_{n=2}^{\infty}\frac{C_n(a)\frac{dC_n(a)}{da}}{n^2-1}-2C_0(a)\frac{dC_0(a)}{da}\right\}. \tag{2.9}$$

Thus, in second approximation, we have

$$x_{II}=a\cos\psi-\frac{\varepsilon C_0(a)}{k}+\frac{\varepsilon}{k}\sum_{n=2}^{\infty}\frac{C_n(a)\cos n\psi}{n^2-1}, \tag{2.10}$$

while

$$\left.\begin{aligned}\frac{da}{dt}&=0,\\ \frac{d\psi}{dt}&=\omega_{II}(a),\end{aligned}\right\} \tag{2.11}$$

where

$$\omega_{II}(a)=\omega+\frac{\varepsilon C_1(a)}{2\omega ma}-\frac{\varepsilon^2}{2\omega}\left[\frac{C_1(a)}{2\omega ma}\right]^2+\frac{\varepsilon^2}{2\omega mka}\left\{\sum_{n=2}^{\infty}\frac{C_n(a)\frac{dC_n(a)}{da}}{n^2-1}-2C_0(a)\frac{dC_0(a)}{da}\right\}. \tag{2.12}$$

We see that also in second approximation, the amplitude a does not depend on time

and preserves its initial value. The phase angle ψ rotates at the constant velocity

$$\psi = \omega_{II}(a)t + \theta \quad (\theta = \text{const}),$$

and eq.(2.10) gives an approximate representation of the general solution (with an accuracy to terms of the order of smallness ε^2), containing two arbitrary integration constants a and θ. We observe that, for conservative oscillatory systems described by an equation of the type of eq.(2.1), all the quantities $A_n(a)$ vanish, since the equation for the amplitude of the fundamental harmonic, with an accuracy to any desired power of ε, will be

$$\frac{da}{dt} = 0,$$

which expresses the condition that the oscillations of arbitrary amplitude must be stationary. Since the error of eq.(2.10) is a quantity of the order of ε^2, then calculations with the same degree of accuracy will yield the following expressions for the maximum and minimum deviations

$$\left.\begin{aligned} x_{II\,max} &= a - \frac{\varepsilon C_0(a)}{k} + \frac{\varepsilon}{k}\sum_{n=2}^{\infty}\frac{C_n(a)}{n^2-1}, \\ x_{II\,min} &= -a - \frac{\varepsilon C_0(a)}{k} + \frac{\varepsilon}{k}\sum_{n=2}^{\infty}\frac{(-1)^n C_n(a)}{n^2-1}. \end{aligned}\right\} \quad (2.13)$$

Before discussing concrete examples, let us transform eq.(2.12), used for determining the dependence of the frequency on the oscillation amplitude.

By squaring both sides and retaining only terms of not more than the second order of smallness, we obtain

$$\left.\begin{aligned} \omega_{II}^2(a) = \omega^2 &+ \frac{\varepsilon C_1(a)}{ma} + \\ &+ \frac{\varepsilon^2}{mka}\left\{\sum_{n=2}^{\infty}\frac{C_n(a)\frac{dC_n(a)}{da}}{n^2-1} - 2C_0(a)\frac{dC_0(a)}{da}\right\}. \end{aligned}\right\} \quad (2.14)$$

Confining ourselves to the first approximation, we have

$$\omega_I^2(a) = \omega^2 + \frac{\varepsilon C_1(a)}{ma}. \quad (2.15)$$

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We note that the linear and nonlinear components of elastic force enter separately in all of the above equations.

The linear component enters by means of the factors ω and k; the nonlinear, by means of the coefficients $C_n(a)$ of the expansion (2.4) for the function $\Phi(a \cos \psi)$.

It is again obvious that the resolution of the total elastic force p(x) of equation (2.2) into a linear and a nonlinear component is, to a considerable extent, arbitrary since the constant k may be selected by various methods.

Let us, for example, determine it from the condition that the "0^{th} approximation" for the oscillation frequency ω coincides with the first approximation $\omega_I(a)$.

Then, eq.(2.15) will yield

$$C_1(a) = 0. \tag{2.16}$$

Let us now consider the Fourier expansion

$$p(a\cos\psi) = p_0(a) + \sum_{n=1}^{\infty} p_n(a)\cos n\psi \tag{2.17}$$

and let us note that eqs.(2.2) and (2.4) give

$$\left.\begin{aligned} p_n(a) &= \varepsilon C_n(a) \quad (n = 0,\ 2,\ 3,\ 4,\ \ldots), \\ p_1(a) &= ak + \varepsilon C_1(a). \end{aligned}\right\} \tag{2.18}$$

Thus the condition (2.16) leads to the following formula for determining the equivalent rigidity:

$$k = \frac{1}{a} p_1(a) = \frac{1}{\pi a}\int_0^{2\pi} p(a\cos\psi)\cos\psi\, d\psi. \tag{2.19}$$

The constant k is determined here as a certain function of the amplitude, and therefore the possibility of using eq.(2.19) is connected with the fact that the amplitude a itself is a constant. If we were considering damped pseudoharmonic oscillations, then such a selection of the constant k would be totally inadmissible, since in that case $\frac{p_1(a)}{a}$ would be a variable quantity with respect to time.

The constant k may also be determined by a different method. If we consider, for example, the expansion in power series of the elastic force in the neighborhood of the point of equilibrium

$$p(x) = \alpha x + \beta x^2 + \gamma x^3 + \dots,$$

it is natural to relate αx to the linear part and the remaining terms to the non-linear

$$\varepsilon\Phi(x) = \beta x^2 + \gamma x^3 + \dots,$$

which corresponds to the selection of k by means of the equation

$$k = p'(0). \tag{2.20}$$

Regardless of what method is used in selecting the constant k, we may, by utilizing eq.(2.18), eliminate the parameter ε from eqs.(2.10), (2.13), (2.14) and (2.15) by rearranging them so that only the known function p(x) will enter.

As a result we reach the following final formulas:

First approximation:

$$\left.\begin{aligned} x_{\mathrm{I}} &= a\cos\psi, \\ \omega_{\mathrm{I}}^2(a) &= \frac{p_1(a)}{ma}. \end{aligned}\right\} \tag{2.21}$$

Second approximation:

$$\left.\begin{aligned} x_{\mathrm{II}} &= a\cos\psi - \frac{p_0(a)}{k} + \frac{1}{k}\sum_{n=2}^{\infty}\frac{p_n(a)\cos n\psi}{n^2-1}, \\ \omega_{\mathrm{II}}^2(a) &= \omega_{\mathrm{I}}^2(a) + \frac{1}{mka}\sum_{n=2}^{\infty}\frac{p_n(a)\dfrac{dp_n(a)}{da}}{n^3-1} - \\ &\qquad 2p_0(a)\frac{dp_0(a)}{da}\frac{1}{mka}. \end{aligned}\right\} \tag{2.22}$$

As is clear, the summand

$$\frac{p_n(a)\dfrac{dp_n(a)}{da}}{mka(n^2-1)}$$

represents the influence of the n^{th} harmonic on the natural frequency, while the summand

$$-2\frac{p_0(a)\dfrac{dp_0(a)}{da}}{mka}$$

is due to the displacement of the working point on the characteristic in connection with the presence of the constant term

$$-\frac{p_0(a)}{k}.$$

in the oscillation.

In the special case of symmetric oscillations, when the elastic charact[...] of the system $F_s = p(x)$ is symmetric* with respect to the origin of coordina[...] that the values of F_s for $\pm x$ are equal in value and opposite in sign

$$p(x) = -p(-x),$$

all even harmonics in the expansion of eq.(2.17) vanish, and the formulas of [...]tion (2.22) assume the form

$$\left.\begin{aligned} x_{II} &= a\cos\psi + \frac{1}{k}\sum_{n=1}^{\infty}\frac{p_{2n+1}(a)\cos(2n+1)\psi}{(2n+1)^2-1}, \\ \omega_{II}^2(a) &= \omega_I^2(a) + \frac{1}{mka}\sum_{n=1}^{\infty}\frac{p_{2n+1}(a)\frac{dp_{2n+1}(a)}{da}}{(2n+1)^2-1}. \end{aligned}\right\} \quad (2.$$

Finally, let us also eliminate the parameter ε from the expressions for [...]mum and minimum deviations.

Equations (2.13) and (2.18) yield

$$\left.\begin{aligned} x_{II\,max} &= a - \frac{p_0(a)}{k} + \frac{1}{k}\sum_{n=2}^{\infty}\frac{p_n(a)}{n^2-1}, \\ x_{II\,min} &= a - \frac{p_0(a)}{k} + \frac{1}{k}\sum_{n=2}^{\infty}\frac{(-1)^n p_n(a)}{n^2-1}. \end{aligned}\right\} \quad (2.$$

To obtain an idea of the practical efficiency of the resultant approxima[...] formulas, let us consider certain numerical examples for which the exact solu[...] known.

Let us consider the equation of the free oscillations of a mathematical [...]lum of mass m and length l without allowing for friction:

* In most cases of practical importance, the elastic force is symmetric. The [...]metry of the curve $F_s = p(x)$ is due, for example, to the action of a const[...] force.

$$m\frac{d^2x}{dt^2} + \frac{g}{l}\sin x = 0, \tag{2.25}$$

where x is the angle of deflection of the pendulum from its equilibrium position.

In this example,

$$p(x) = \frac{g}{l}\sin x$$

so that the expansion of eq.(2.17) will be

$$p(a\cos\psi) = \frac{g}{l}\,2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(a)\cos(2n+1)\psi.$$

[$J_k(a)$ are Bessel functions].

According to eqs.(2.21) and (2.23) we find

in first approximation:

$$\left.\begin{aligned} x_I &= a\cos\psi, \\ \left(\frac{\omega_I}{\omega_0}\right)^2 &= \frac{2J_1(a)}{a} \end{aligned}\right\} \tag{2.26}$$

where $\omega_0^2 = \frac{g}{lm}$;

in second approximation:

$$\left.\begin{aligned} x_{II} &= a\cos\psi + 2\frac{\omega_0^2}{\omega_I^2}\sum_{n=1}^{\infty}\frac{(-1)^n J_{2n+1}(a)\cos(2n+1)\psi}{(2n+1)^2-1}, \\ \left(\frac{\omega_{II}}{\omega_0}\right)^2 &= \left(\frac{\omega_I}{\omega_0}\right)^2 + 4\frac{\omega_0^2}{\omega_I^2 a}\sum_{n=1}^{\infty}\frac{J_{2n+1}(a)\cdot J'_{2n+1}(a)}{(2n+1)^2-1}. \end{aligned}\right\} \tag{2.27}$$

In particular:

$$x_{I\,max} = a, \quad x_{II\,max} = a + 2\frac{\omega_0^2}{\omega_I^2}\sum_{n=1}^{\infty}\frac{(-1)^n J_{2n+1}(a)}{(2n+1)^2-1}. \tag{2.28}$$

On considering x_{max} and ω as functions of the amplitude of the first harmonic

$$x_{max} = x_{max}(a), \quad \omega = \omega(a), \tag{2.29}$$

we can calculate their approximate values for a number of possible values of a.

In view of the very rapid convergence of the series in the right-hand sides of the expressions (2.27) and (2.28), it is sufficient to take only their first two

terms into account in the calculation.

Moreover, to evaluate the accuracy of the solutions (2.28) obtained, let us calculate from Tables of elliptic functions, the corresponding values of eq. (2.29) by the aid of the exact formulas (for the same values of a):

$$\left.\begin{aligned} x(\varphi, a) &= \frac{8\sqrt{q}}{1+q}\cos\varphi - \frac{8q^{3/2}}{3(1+q^3)}\cos 3\varphi + \ldots, \\ \frac{\omega}{\omega_0} &= \frac{\pi}{2K'} \\ k &= \sin\frac{x_{max}}{2}; \end{aligned}\right\} \quad (2.30)$$

Here, as is customary in manuals on the theory of elliptic functions, k is the modulus, K the total elliptic integral of the first kind; $q = e^{-\pi\frac{K'}{K}}$; while $K'(k) = K(k')$, where $k' = \sqrt{1-k^2}$.

Table 1 gives the results of the calculations, and contains a column giving the exact x_{max} in degrees.

The results of the calculations indicate completely satisfactory accuracy, particularly if we bear in mind that our approximate formulas are derived under the assumption that the elastic characteristic of the restoring force has a "weak" nonlinearity, close to linear. In the example under consideration, even at angles of deflection of the pendulum amounting to about 160°, the relative error of the first approximation of frequency is 5.5%, and of the second only about 3%, although it is obvious that, in the range from - 160° to + 160°, the sine is a very poor approximation to a straight line. At oscillations of the pendulum from about - 30° to + 30°, the first approximation gives four correct decimal places, while for angles between ± 45°, the second approximation is correct to five places of decimals. Consequently, where the characteristic is actually close to linear, the resultant approximate formulas possess a high degree of accuracy.

The impairment of accuracy for angles close to 180° is explained by the fact that this value is critical since, on transition through it, the character of the motion is changed, and oscillation is replaced by rotation.

Let us discuss studies of the small oscillations of a pendulum. In this case,

sine x in eq.(2.25) can be replaced by the two or three first terms of the Taylor expansion* (depending on the approximation at which we intend to stop).

Table 1

$x_{I\,max} = a$	$x_{II\,max}$	x_{max}	$\frac{\omega_I}{\omega_0}$	$\frac{\omega_{II}}{\omega_0}$	$\frac{\omega}{\omega_0}$	x''_{max}
0,2	0,19996	0,19996	0,99751	0,99751	0,99751	11°27'25"
0,4	0,39966	0,39968	0,99002	0,99003	0,99003	22°53'46"
0,6	0,5988	0,5989	0,97759	0,97763	0,97762	34°18'52"
0,8	0,7972	0,7973	0,9602	0,96040	0,96040	15°40'55"
1,0	0,9944	0,9946	0,9381	0,93847	0,93840	56°59'11"
1,2	1,1900	1,1906	0,9113	0,91201	0,91198	68°12'59"
1,4	1,3835	1,3846	0,8799	0,88122	0,88114	79°19'51"
1,6	1,5743	1,5763	0,844	0,8463	0,8461	90°18'55"
1,8	1,761	1,765	0,804	0,8076	0,8072	101°07'37"
2,0	1,943	1,951	0,759	0,7654	0,7646	111°47'03"
2,2	2,118	2,132	0,711	0,7200	0,7185	122°09'17"
2,4	2,283	2,307	0,658	0,6719	0,6698	132°10'53"
2,6	2,432	2,476	0,602	0,6216	0,6138	141°51'52"
2,8	2,558	2,635	0,541	0,5699	0,5610	150°58'28"
3,0	2,642	2,783	0,475	0,5179	0,5023	159°27'15"

We then obtain

$$m\frac{d^2x}{dt^2} + \frac{g}{l}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) = 0. \qquad (2.31)$$

Applying eq.(2.21) to this equation, we find (confining ourselves to the first two terms in the expansion of the sine):

* We note here that the difference

$$\sin x - \left(x - \frac{x^3}{3!}\right)$$

does not exceed 0.000326 in absolute value, if x oscillates between - 30 and + 30°, while the difference

$$\sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)$$

does not exceed 0.000002.

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$$x_I = a\cos\psi, \qquad (2.32)$$
$$\frac{\omega_I^2(a)}{\omega_0^2} = 1 - \frac{a^2}{8},$$

whence

$$\frac{\omega_I(a)}{\omega_0} = \sqrt{1 - \frac{a^2}{8}} \approx 1 - \frac{a^2}{16}. \qquad (2.33)$$

It is obvious from eq.(2.33) that an increase in the amplitude of the oscillation of the pendulum causes a decrease in the frequency, while the period of the natural oscillations

$$T_I = \frac{T_0}{1 - \frac{a^2}{16}} \approx T_0\left(1 + \frac{a^2}{16}\right) \qquad (2.34)$$

increases. (Here $T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{lm}{g}}$).

To construct the solution in second approximation, let us utilize eq.(2.23); then, taking account in the expansion for sin x of the term $\frac{x^5}{5!}$ we obtain

$$\left.\begin{aligned} x_{II} &= a\cos\psi - \frac{a^3}{192}\left(1 + \frac{3}{64}a^2\right)\cos 3\psi + \frac{a^5}{20480}\cos 5\psi, \\ \frac{\omega_{II}^2(a)}{\omega_0^2} &= 1 - \frac{a^2}{8} + \frac{3a^4}{512}. \end{aligned}\right\} \qquad (2.35)$$

whence

$$\frac{\omega_{II}(a)}{\omega_0} \approx 1 - \frac{a^2}{16} + \frac{a^4}{1024} \qquad (2.36)$$

and

$$T_{II} = T_0\left(1 + \frac{a^2}{16} + \frac{3a^4}{1024}\right). \qquad (2.37)$$

For the maximum deviations, eq.(2.24) yields

$$x_{II\,max} = a - \frac{a^3}{192} - \frac{a^5}{1024}. \qquad (2.38)$$

The resultant formulas can be used for calculating the frequencies, periods, and maximum deviations for a number of values of a (amplitudes of the first har-

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monic). Table 2 gives the results of these calculations.

Table 2

$x_{I\,max} = a$	$x_{II\,max}$	$\frac{\omega_I}{\omega_0}$	$\frac{\omega_{II}}{\omega_0}$	$\frac{T_I}{T_0}$	$\frac{T_{II}}{T_0}$
0,2	0,19996	0,99750	0,99750	1,00250	1,00250
0,4	0,39966	0,99000	0,99003	1,01000	1,01008
0,6	0,5988	0,97750	0,97763	1,02250	1,02288
0,8	0,7970	0,9600	0,96010	1,0400	1,04120
1,0	0,9938	0,9375	0,93848	1,0625	1,06543
1,2	1,1886	0,9100	0,91203	1,0900	1,09607
1,4	1,3805	0,8775	0,88125	1,1225	1,13376
1,6	1,5684	0,840	0,8464	1,160	1,1792
1,8	1,751	0,798	0,8078	1,203	1,2333
2,0	1,927	0,750	0,7656	1,250	1,2969
2,2	2,094	0,698	0,7204	1,303	1,3711
2,4	2,250	0,640	0,6724	1,360	1,4572
2,6	2,392	0,578	0,6221	1,423	1,5561
2,8	2,518	0,510	0,5700	1,490	1,6701
3,0	2,622	0,438	0,5166	1,563	1,7998

A comparison of Table 2 with Table 1 shows readily that, for deflections of the pendulum not exceeding ± 35° (within these limits, the frequencies and maximum deviations agree with the exact values to the fifth and fourth decimal place inclusive), we may successfully use eq.(2.31) and the corresponding simpler approximate solutions (2.32) and (2.35) instead of the exact equation (2.25). At greater angles of deflection, of the order of ± 160°, the relative error of the first approximation amounts to 13% and of the second, to only about 3%.

In considering the free oscillations of the pendulum, we have disregarded the forces of friction.

Assuming that the oscillations of the pendulum are damped under the action of forces proportional to the velocity, we arrive at the investigation of the following equation:

$$m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + \frac{g}{l}\sin x = 0, \tag{2.39}$$

or, for small deflections,

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$$m \frac{d^2x}{dt^2} + \lambda \frac{dx}{dt} + \frac{g}{l}\left(x - \frac{x^3}{6}\right) = 0. \qquad (2.40)$$

According to the general formula of Section 1, the solution of eq.(2.40) will be in first approximation

$$x = a \cos \psi, \qquad (2.41)$$

where a and ψ must be determined from the system of equations of the first approximation

$$\left.\begin{aligned} \frac{da}{dt} &= -\delta a, \\ \frac{d\psi}{dt} &= \omega\left(1 - \frac{a^2}{16}\right), \end{aligned}\right\} \qquad (2.42)$$

where we have introduced the symbol $\delta = \frac{\lambda}{2m}$, $\omega = \sqrt{\frac{g}{lm}}$.

On integrating the first equation of the system (2.42) at the initial values of $t = 0$, $a = a_0$, we find

$$a = a_0 e^{-\delta t}. \qquad (2.43)$$

After this, the second equation of the system (2.42) yields

$$\psi = \omega\left\{t + \frac{a_0^2}{32\delta}\left(e^{-2\delta t} - 1\right)\right\} + \theta, \qquad (2.44)$$

where θ is the initial value of the phase.

On substituting the values of the amplitude (2.43), and of the phase (2.44) in eq.(2.41), we obtain the first approximation in the form:

$$x = a_0 e^{-\delta t} \cos\left\{\omega\left[t + \frac{a^2}{32\delta}\left(e^{-2\delta t} - 1\right)\right] + \theta\right\}. \qquad (2.45)$$

Thus, in first approximation, the oscillations will be damped, and will have a frequency depending on the amplitude $\omega = \omega(a)$; with increasing time, the gradual damping will cause the instantaneous frequency to increase, approaching, as a limit, the constant "linear" value of the frequency $\omega = \sqrt{\frac{g}{lm}}$.

Let us now consider the oscillations of a system in which the characteristic of the restoring elastic force has the form

$$p(x) = \alpha x + \gamma x^3, \quad (\alpha > 0, \ \gamma > 0). \qquad (2.46)$$

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In this case, we obtain the nonlinear differential equation

$$m\frac{d^2x}{dt^2} + \alpha x + \gamma x^3 = 0, \tag{2.47}$$

which may be integrated in explicit form by elliptic functions. Consequently, here too the approximate solutions may be compared with the exact solutions.

Since, for this case, the expansion (2.17) will be

$$f(a\cos\psi) = \left(\alpha a + \frac{3}{4}\gamma a^3\right)\cos\psi + \frac{\gamma a^3}{4}\cos 3\psi,$$

the introduction of the dimensionless combinations $\left(\frac{\gamma}{\alpha}\right)^{1/2} x$, $\left(\frac{\gamma}{\alpha}\right)^{1/2} a$, $\frac{\omega}{\omega_0}$ and the use of eqs. (2.21) and (2.23) will give

in first approximation:

$$\left.\begin{aligned} &\left(\frac{\gamma}{\alpha}\right)^{1/2} x_I = \left(\frac{\gamma}{\alpha}\right)^{1/2} a\cos\psi, \\ &\left(\frac{\omega_I(a)}{\omega_0}\right)^2 = 1 + \frac{3}{4}\left[\left(\frac{\gamma}{\alpha}\right)^{1/2} a\right]^2, \quad \omega_0 = \sqrt{\frac{\alpha}{m}}, \end{aligned}\right\} \tag{2.48}$$

in second approximation:

$$\left.\begin{aligned} &\left(\frac{\gamma}{\alpha}\right)^{1/2} x_{II} = \left(\frac{\gamma}{\alpha}\right)^{1/2} a\cos\psi + \frac{\left(a\sqrt{\frac{\gamma}{\alpha}}\right)^3\cos 3\psi}{32\left(\frac{\omega_I}{\omega_0}\right)^2}, \\ &\left(\frac{\omega_{II}(a)}{\omega_0}\right)^2 = \left(\frac{\omega_I(a)}{\omega_0}\right)^2\left\{1 + \frac{3}{128}\frac{\left(a\sqrt{\frac{\gamma}{\alpha}}\right)^4}{\left(\frac{\omega_I}{\omega_0}\right)^4}\right\}, \end{aligned}\right\} \tag{2.49}$$

whence we have

$$\left(\frac{\gamma}{\alpha}\right)^{1/2} x_{I\max} = \left(\frac{\gamma}{\alpha}\right)^{1/2} a; \quad \left(\frac{\gamma}{\alpha}\right)^{1/2} x_{II\max} = \left(\frac{\gamma}{\alpha}\right)^{1/2} a\left\{1 + \frac{a^2\left(\frac{\gamma}{\alpha}\right)}{32\left(\frac{\omega_I}{\omega_0}\right)^2}\right\}. \tag{2.50}$$

We find the exact values of $\left(\frac{\gamma}{\alpha}\right)^{1/2} x_{max}$ and $\frac{\omega}{\omega_0}$ for given values of $\left(\frac{\gamma}{\alpha}\right)^{1/2} a$ from Tables of elliptic functions, by using the formula

$$x(\psi, a) = x_{\max}\cdot \mathrm{cn}\left\{\frac{2K}{\pi}\psi\right\} =$$

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$$= x_{max}\frac{2\pi}{kK}\left\{\frac{\sqrt{q}}{1+q}\cos\psi + \frac{q^{3/2}}{1+q^{3}}\cos 3\psi + \dots\right\}, \tag{2.51}$$

$$\frac{\omega}{\omega_0} = \frac{\pi\sqrt{1+\xi^2}}{2K}, \quad k = \frac{\xi}{\sqrt{2+2\xi^2}}, \quad \xi = x_{max}\left(\frac{\gamma}{\alpha}\right)^{1/2}.$$

Here cn, k, K, and q denote respectively the elliptic cosine, the modulus, the total elliptic interval of the first kind, and $e^{-\pi\frac{K'}{K}}$, while $K'(k) = K(k')$, where $k' = \sqrt{1-k^2}$.

The results of the calculations are given in Table 3.

Table 3

$\left(\frac{\gamma}{\alpha}\right)^{1/2} a$	$\left(\frac{\gamma}{\alpha}\right)^{1/2} x_{II\,max}$	$\left(\frac{\gamma}{\alpha}\right)^{1/2} x_{max}$	$\frac{\omega_I}{\omega_0}$	$\frac{\omega_{II}}{\omega_0}$	$\frac{\omega}{\omega_0}$
0,29927	0,30005	0,3	1,0330	1,0331	1,0331
0,59464	0,5998	0,6	1,1248	1,1258	1,1259
0,88552	0,8992	0,9	1,2602	1,2638	1,2641
1,1733	1,1981	1,2	1,4256	1,4333	1,4340
1,4592	1,4966	1,5	1,6115	1,6241	1,6257
1,7443	1,7948	1,8	1,8116	1,8297	1,8323
2,0293	2,0931	2,1	2,022	2,0459	2,0493
2,3140	2,3912	2,4	2,240	2,2697	2,2740
2,5991	2,6895	2,7	2,463	2,4985	2,5041
2,8811	2,9877	3,0	2,690	2,7318	2,7385

Bearing in mind the simplicity of eqs.(2.48) and (2.49) we must also recognize in this example that the degree of approximation obtained is entirely satisfactory. It can also be shown that these formulas are still valid even when $a\left(\frac{\gamma}{\alpha}\right)^{1/2} \to \infty$.

Equation (2.51) leads to the following asymptotic formula

$$\frac{\omega}{\omega_0} = \frac{1}{4\sqrt{2}}\,\frac{1-q}{1+q}\,a\left(\frac{\gamma}{\alpha}\right)^{1/2} + \dots,$$

where q is taken for the modulus $k = \frac{1}{\sqrt{2}}$, while the dots denote a term whose ratio to the above first term approaches zero as $a\left(\frac{\gamma}{\alpha}\right)^{1/2} \to \infty$. Analogous asymptotic formulas are obtained from eqs.(2.48) and (2.49) for $\frac{\omega_I(a)}{\omega_0}$ and $\frac{\omega_{II}(a)}{\omega_0}$, respectively, with factors of proportionality of $\sqrt{\frac{3}{4}}$, $\sqrt{\frac{3}{4}}\left(1 + \frac{3}{128}\left(\frac{4}{3}\right)^2\right)$. Their numerical value will

be

$$\frac{1}{4\sqrt{2}}\left(\frac{1+q}{\sqrt{q}}\right)_{q=\frac{1}{\sqrt{3}}} = 0,887, \quad \sqrt{\frac{3}{4}} = 0,866;$$

$$\sqrt{\frac{3}{4}}\left(1+\frac{3}{128}\left(\frac{4}{3}\right)^2\right) = 0,892.$$

Thus, at the limit when a $\left(\frac{\gamma}{\alpha}\right)^{1/2} \to \infty$, the relative error of the first approximation of frequency amounts to 2.4%, and that of the second approximation to only 0.6%.

Section 3. The Case of Nonlinear Friction

As a second special case, let us consider an equation of the form

$$m\frac{d^2x}{dt^2} + kx = \epsilon F\left(\frac{dx}{dt}\right), \tag{3.1}$$

which may be interpreted as the equations of oscillation of the mass m under the action of the linear elastic force kx and the nonlinear weak friction $\epsilon F\left(\frac{dx}{dt}\right)$, depending on the velocity.

This equation obviously belongs to the type of the general eq.(1.1), while here

$$f\left(x, \frac{dx}{dt}\right) = \frac{1}{m} F\left(\frac{dx}{dt}\right).$$

In order to make use of eq.(1.21)-(1.28) for determining the wanted approximate solutions, let us consider the expansion

$$\frac{1}{m} F(a\cos\psi) = \sum_{n=0}^{\infty} F_n(a)\cos n\psi, \tag{3.2}$$

from which we obtain $\frac{1}{m} F(-a\omega\sin\psi) = \sum_{n=0}^{\infty} F_n(a\omega)\cos n\left(\psi + \frac{\pi}{2}\right).$

On comparing the latter expansion with eq.(1.16), we find

$$g_n(a) = F_n(a\omega)\cos\frac{n\pi}{2}, \quad h_n(a) = -F_n(a\omega)\sin\frac{n\pi}{2}. \tag{3.3}$$

For this reason, eq.(1.17) will yield

$$A_1(a) = -\frac{1}{2\omega} F_1(a\omega), \quad B_1(a) = 0. \tag{3.4}$$

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Thus, taking eqs.(1.23) and (1.24) into account, we obtain the first approximation in the following form:

$$\left.\begin{aligned} x &= a\cos\psi, \\ \frac{da}{dt} &= \frac{\varepsilon}{2\omega} F_1(a\omega), \\ \frac{d\psi}{dt} &= \omega, \quad \omega = \sqrt{\frac{k}{m}}. \end{aligned}\right\} \tag{3.5}$$

This readily shows that, for these systems described by an equation of the type of eq.(3.1), the amplitude of oscillation in first approximation is damped by a law expressed by the first equation of the system (3.5). As for the instantaneous frequency, it is constant and equal to the ordinary linear frequency ω, so that

$$\psi = \omega t + \theta,$$

where θ is the initial value of the phase ψ.

Thus, in first approximation, the oscillations are found to be harmonic, at a constant frequency ω.

We have already had an opportunity to prove that nonlinear oscillatory systems, generally speaking, are not isochronous.

The example under consideration, however, is one of the important cases when, in first approximation, the system is isochronous. Such cases will be denoted as quasi-isochronous*.

Let us discuss the construction of the second approximation: Equations (1.28) and (3.3) yield

$$u_1(a,\psi) = -\frac{1}{\omega^2}\sum_{\substack{n=0 \\ (n\neq 1)}}^{\infty} \frac{F_n(a\omega)\cos n\left(\psi+\frac{\pi}{2}\right)}{n^2-1}. \tag{3.6}$$

From eq.(1.30), (3.4) and (3.6), we obtain

$$A_2(a) = -\frac{A_1(a)}{2\omega\pi m}\int_0^{2\pi} F'(-a\omega\sin\psi)\cos\psi\sin\psi\, d\psi -$$

* We add the "quasi" because the corresponding oscillations of the system, as will be shown below, will be isochronous only in first approximation.

$$
\left.
\begin{aligned}
& - \frac{1}{2\omega^2 \pi m} \sum_{\substack{n=0 \\ (n \neq 1)}}^{\infty} \frac{nF_n(a\omega)}{n^2 - 1} \int_0^{2\pi} F'(-a\omega \sin\psi) \times \\
& \qquad \times \sin n\left(\psi + \frac{\pi}{2}\right) \sin\psi \, d\psi, \\
& \times \cos^2\psi \, d\psi - \frac{1}{2\omega^2 \pi a m} \sum_{\substack{n=0 \\ (n \neq 1)}}^{\infty} \frac{nF_n(a\omega)}{n^2 - 1} \times \\
& B_2(a) = \frac{1}{2\omega} \frac{A_1(a)}{a} \frac{dA_1(a)}{da} - \frac{A_1(a)}{2\omega\pi a m} \int_0^{2\pi} F'(-a\omega \sin\psi) \times \\
& \qquad \times \int_0^{2\pi} F'(-a\omega \sin\psi) \sin n\left(\psi + \frac{\pi}{2}\right) \cos\psi \, d\psi.
\end{aligned}
\right\} \quad (3.7)
$$

On the other hand, a substitution of $\psi - \frac{\pi}{2}$ for ψ in the integrals, gives

$$
\int_0^{2\pi} F'(-a\omega \sin\psi) \sin n\left(\psi + \frac{\pi}{2}\right) \sin\psi \, d\psi =
$$

$$
= - \int_0^{2\pi} F'(a\omega \cos\psi) \sin n\psi \cos\psi \, d\psi = 0.
$$

Further, integrating by parts, we get

$$
\frac{1}{m} \int_0^{2\pi} F'(-a\omega \sin\psi) \sin n\left(\psi + \frac{\pi}{2}\right) \cos\psi \, d\psi =
$$

$$
= \frac{1}{m} \int_0^{2\pi} F'(a\omega \cos\psi) \sin n\psi \sin\psi \, d\psi = -\frac{1}{m} \int_0^{2\pi} \sin n\psi \, d\left[\frac{F(a\omega \cos\psi)}{a\omega}\right] =
$$

$$
= \frac{n}{a\omega m} \int_0^{2\pi} F(a\omega \cos\psi) \cos n\psi \, d\psi = \frac{n\pi}{a\omega} F_n(a\omega).
$$

Equation (3.7) may thus be written as follows:

$$
A_2(a) = 0,
$$

$$
B_2(a) = \frac{F_1(a\omega)}{8\omega^3 a} \frac{dF_1(a\omega)}{da} - \frac{F_1^2(a\omega)}{4\omega^3 a^2} - \frac{1}{2\omega^3 a^2} \sum_{n=2}^{\infty} \frac{n^2 F_n^2(a\omega)}{n^2 - 1}. \quad (3.8)
$$

In this case, the second approximation has the form

$$
x = a\cos\psi - \frac{\varepsilon}{\omega^2} \sum_{\substack{n=0 \\ (n \neq 1)}}^{\infty} \frac{F_n(a\omega)\cos\left(\psi + \frac{\pi}{2}\right)}{n^2 - 1}, \quad (3.9)
$$

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while

$$\left.\begin{aligned} \frac{da}{dt} &= \frac{\varepsilon F_1(a\omega)}{2\omega}, \\ \frac{d\psi}{dt} &= \omega + \varepsilon^2 B_2(a), \end{aligned}\right\} \qquad (3.10)$$

where $B_2(a)$ is determined from eq.(3.8).

Before discussing the analysis of the equations for the amplitude as a function of time for various laws of the force of friction, i.e., for various forms of the function $F\left(\frac{dx}{dt}\right)$, we note that, for these formulas to be applicable, a general limitation is necessary, namely that the force of friction must be sufficiently small.

In describing the analysis of concrete examples, let us first consider the linear equation

$$\frac{d^2x}{dt^2} + \lambda \frac{dx}{dt} + \omega^2 x = 0 \qquad (3.11)$$

with the small damping coefficient

$$\lambda = \varepsilon\nu.$$

For this equation

$$F\left(\frac{dx}{dt}\right) = -\nu \frac{dx}{dt},$$

and, therefore,

$$F_1(a\omega) = -\nu\omega a,$$

$$F_n(a\omega) = 0 \qquad (n = 0, 2, 3, \ldots).$$

Thus, eqs.(3.8), (3.9), and (3.10) yield directly, in second approximation,

$$x = a\cos\psi,$$

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\lambda a}{2}, \\ \frac{d\psi}{dt} &= \omega\left\{1 - \frac{1}{8}\left(\frac{\lambda}{\omega}\right)^2\right\}. \end{aligned}\right\} \qquad (3.12)$$

As indicated by the first equation of the system (3.12), full agreement is obtained between the law of damping of the amplitude and the exact formula

$$a = a_0 e^{-\frac{\lambda t}{2}},$$

while, for the oscillation frequency, we have the approximate formula

$$\omega_2 = \omega\left\{1 - \frac{1}{8}\left(\frac{\lambda}{\omega}\right)^2\right\}, \tag{3.13}$$

which corresponds to the first two summands in the expansion of the exact expression for the frequency

$$\omega\sqrt{1 - \frac{1}{4}\left(\frac{\lambda}{\omega}\right)^2} = \omega\left\{1 - \frac{1}{8}\left(\frac{\lambda}{\omega}\right)^2 - \frac{1}{128}\left(\frac{\lambda}{\omega}\right)^4 + \dots\right\}$$

in powers of $\frac{\lambda}{\omega}$, which, however is entirely natural, since we are disregarding terms of an order of smallness higher than the second.

To get an idea of the degree of accuracy of the resultant approximate eq.(3.13), let us take, for example, $\frac{\lambda}{\omega} = \frac{\ln 2}{\pi}$. We note that this value of the coefficient λ corresponds to a considerable damping. Thus, in one period, the amplitude of the oscillations diminishes to half. In absolute value, the "perturbation term" $\lambda\frac{dx}{dt}$ still is about $\frac{1}{4}$ of the "principal terms" $\frac{d^2x}{dt^2}$ or $\omega^2 x$. In spite of this, the relative error of eq.(3.3) is less than 0.01%.

Let us consider another simple example leading to an equation of the type of eq.(3.1) namely a harmonic, or any small oscillations of a pendulum in a medium whose resistance is proportional to the square of the velocity and is small.

In this case, the equation of the oscillation will be

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \alpha\left(\frac{dx}{dt}\right)^2 + \omega^2 x = 0, \quad \text{если}\ \frac{dx}{dt} > 0, \\ \frac{d^2x}{dt^2} - \alpha\left(\frac{dx}{dt}\right)^2 + \omega^2 x = 0, \quad \text{если}\ \frac{dx}{dt} < 0 \end{aligned}\right\} \tag{3.14}$$

or

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt}\left|\frac{dx}{dt}\right| + \omega^2 x = 0, \tag{3.15}$$

where, as always $\left|\frac{dx}{dt}\right|$* denotes the absolute value of $\frac{dx}{dt}$. (We resort to this notation to indicate that the term $\alpha\left(\frac{dx}{dt}\right)^2$ represents a resistance to motion).

* Translator's note: See errata sheet.

Considering the damping sufficiently weak, we use

$$\gamma = 2\epsilon.$$

Then eq.(3.15) will be an equation of the form of eq.(3.1), where

$$F\left(\frac{dx}{dt}\right) = -\gamma \frac{dx}{dt}\left|\frac{dx}{dt}\right|.$$

Let us find the expression of the n^th term in the Fourier expansion for F(a cos ψ):

$$F_n(a) = \frac{2}{\pi}\int_0^{\pi} F(a\cos\psi)\cos n\psi \, d\psi =$$

$$= \frac{2\gamma a^2}{\pi}\int_0^{\pi} |\cos\psi| \cos\psi \cos n\psi \, d\psi =$$

$$= \frac{2\gamma a^2}{\pi}\left\{\int_0^{\frac{\pi}{2}} \cos^2\psi \cos n\psi \, d\psi - \int_{\frac{\pi}{2}}^{\pi} \cos^2\psi \cos n\psi \, d\psi\right\},$$

whence

$$F_0(a) = F_2(a) = F_4(a) = \ldots = F_{2q}(a) = \ldots = 0,$$

$$F_1(a) = \frac{8\gamma a^2}{3\pi}, \quad F_{2q+1}(a) = \frac{8\gamma a^2(-1)^{q+1}}{\pi(2q+1)[(2q+1)^2-4]}$$

$$(q = 0, 1, 2, \ldots).$$

Thus, eqs.(3.8), (3.9), and (3.10) will give the second approximation in the form

$$x = a\cos\psi - \frac{8\gamma a^2}{\pi}\sum_{q=1}^{\infty}\frac{\sin(2q+1)\psi}{(2q+1)[(2q+1)^2-1][(2q+1)^2-4]} = \quad \textbf{(3.16)}$$

$$= a\cos\psi - \frac{\gamma a^2}{15\pi}\left\{\sin 3\psi + \frac{1}{21}\sin 5\psi + \ldots\right\}, \quad (3.16)$$

where a and ψ are determined by the equations

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{4\gamma\omega}{3\pi}a^2, \\ \frac{d\psi}{dt} &= \omega\left\{1 - \frac{4\gamma^2 a^2}{\pi^2}C\right\}; \end{aligned}\right\} \quad \textbf{(3.17)}$$

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here, for abbreviation, we introduce the symbols

$$C = 8\sum_{q=1}^{\infty} \frac{1}{[(2q+1)^2-1][(2q+1)^2-4]^2} \tag{3.18}$$

$$= \frac{1}{25} + \frac{1}{1323} + \frac{1}{12150} + \ldots = 0{,}0407\ldots$$

By integrating the first equation of the system (3.17), we have

$$\frac{1}{a} - \frac{1}{a_0} = \frac{4\alpha\omega}{3\pi}t, \tag{3.19}$$

whence we find the law of damping of the amplitude of the fundamental harmonic of the oscillation:

$$a = \frac{a_0}{1 + \frac{4\alpha\omega a_0}{3\pi}t}. \tag{3.20}$$

In this way, the amplitude of the oscillations, for the square-law of damping, is damped approximately inversely proportional to the increase of the linear function of time.

On substituting eq.(3.20) in the second of the equations (3.17), and integrating, we obtain the law of rotation of the phase angle

$$\psi = \omega t - \frac{3C\alpha a_0}{\pi}\left\{1 - \frac{1}{1 + \frac{4\alpha\omega a_0}{3\pi}t}\right\} + \psi_0. \tag{3.21}$$

Thus, we have explicit expressions for the representation of the oscillatory process, in second approximation.

We note that the correction terms of the second approximation are very small, even at considerable damping. Thus, if we take $\alpha a_0 = \frac{3}{8}$, i.e., if we consider the case in which the amplitude a, one cycle after the beginning of the oscillations, is reduced to one-half, then the sum of the amplitudes of all first harmonics of the oscillations will amount to less than 1% of the amplitude of the fundamental harmonic; but the correction of the second approximation for the frequency of the oscillations will be less than 0.25%.

Let us now compare the resultant approximate solution with the exact solution:

Although eq.(3.15) can be integrated to the end, as a result, we would arrive at the transcendental quadrature

$$\int_{z_0}^{z} \frac{dz}{\sqrt{(1+2\gamma z)-(1+2\gamma z_0)e^{-2\gamma(z-z_0)}}} = \frac{\omega}{2\gamma}t \qquad (3.22)$$

(where z denotes the distance between the extreme positions of the oscillating pendulum) and thus the function required cannot be represented by the aid of the elementary functions. It is not, however, difficult to establish the equation for two successive amplitudes damped by the presence of a friction proportional to the square of the velocity. In accordance with F.Prasil* we have

$$(2\gamma z_I + 1) - \ln(2\gamma z_I + 1) = (2\gamma z_{II} - 1) - \ln(2\gamma z_{II} - 1) \qquad (3.23)$$

or, in our notation,

$$(4\gamma a_0 + 1) - \ln(4\gamma a_0 + 1) = (4\gamma a_1 + 1) - \ln(4\gamma a_1 - 1), \qquad (3.24)$$

where a_0 is the initial value of amplitude and a_1 the value of the amplitude after one period of oscillation has elapsed.

In order to compare the results obtained by the exact formula (3.24) with those of the approximate formula (3.19), let us transform eq.(3.19). It may obviously be represented in the following form:

$$\frac{1}{4\gamma a} - \frac{1}{4\gamma a_0} = \frac{\omega}{3\pi}t. \qquad (3.25)$$

On substituting, in the right-hand side, the value of the period in first approximation, we obtain the following relation which connects two successive amplitudes**:

$$\frac{1}{4\gamma a_1} - \frac{1}{4\gamma a_0} = \frac{2}{3}. \qquad (3.26)$$

The Table given below shows the good agreement of the successive amplitudes calculated by the exact eq.(3.24) and the approximate formula. For $4\alpha a_0 = 1$, i.e.,

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* F.Prasil (Bibl.46)

** We note that the same approximate formula was empirically found by A.de Caligny, (Bibl.47)

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for the case when the amplitude decreases to 0.6 of its value after one cycle, the results obtained from eq.(3.26) (which characterizes only the first approximation)

Table 4

$(4\alpha a)_{exact}$	$(4\alpha a)_{app}$	$(4\alpha a)_{ex}$	$(4\alpha a)_{app}$	$(4\alpha a)_{ex}$	$(4\alpha a)_{app}$
1,0000	1,0000	0,1570	0,1578	0,0854	0,0856
0,5936	0,6000	0,1420	0,1428	0,0808	0,0810
0,4240	0,4285	0,1298	0,1304	0,0767	0,0769
0,3301	0,3332	0,1191	0,1200	0,0730	0,0731
0,2704	0,2726	0,1106	0,1111		
0,2290	0,2307	0,1030	0,1034		
0,1986	0,1999	0,0964	0,0967		
0,1753	0,1764	0,0906	0,0908		

differ by only 1% from the exact results of eq.(3.24). For the case of $4\alpha a_0 = 0.1$, however, it differs only by 0.4%.

Section 4. Self-Sustained Oscillatory Systems

Let us consider another oscillatory system, which is described by an equation of the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f(x) \frac{dx}{dt}, \qquad (4.1)$$

which also is a special case of eq.(1.1). We note that eq.(3.1), previously considered, may be reduced to the form of eq.(4.1).

Now, putting

$$\frac{dx}{dt} = y$$

and differentiating eq.(3.1), we obtain

$$m \frac{d^2y}{dt^2} + ky = \varepsilon F'(y) \frac{dy}{dt}.$$

On comparing eq.(4.1) with eq.(1.1), we have

$$f\left(x, \frac{dx}{dt}\right) = f(x)\frac{dx}{dt};$$

Therefore, in order to make use of eqs.(1.21)-(1.28), the expression

$$f(a\cos\psi)\,a\omega\sin\psi.$$

must be expanded into a Fourier series.

To simplify this operation, let us consider the function

$$F^*(x) = \int_0^x f(x)\,dx \tag{4.2}$$

and its expansion into a Fourier series

$$F^*(a\cos\psi) = \sum_{n=0}^{\infty} F_n^*(a)\cos n\psi. \tag{4.3}$$

On differentiating eq.(4.3) with respect to ψ, we obtain, on the basis of equation (4.2),

$$f(a\cos\psi)\,a\omega\sin\psi = \sum_{n=0}^{\infty} \omega n F_n^*(a)\sin n\psi. \tag{4.4}$$

By comparing eq.(4.4) with eqs.(1.16) and (1.17), we find

$$A_1(a) = \frac{1}{2}F_1^*(a), \quad B_1(a) = 0, \tag{4.5}$$

whence, in first approximation, we have

$$x = a\cos\psi,$$

where a and ψ must satisfy the equations

$$\left.\begin{aligned} \frac{da}{dt} &= \frac{\varepsilon}{2}F_1^*(a), \\ \frac{d\psi}{dt} &= \omega, \end{aligned}\right\} \tag{4.6}$$

The results of the preceding Section may be used in constructing the second approximation.

Starting from eqs.(3.2), (3.6), (3.7), (3.8) and (4.4), and bearing in mind that

$$\frac{1}{\pi}\int_0^{\pi} f(a\cos\psi)\cos\psi\sin\psi\,d\psi = 0,$$

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$$\frac{1}{\pi}\int_0^{\pi} f(a\cos\psi)\cos^2\psi\, d\psi = \frac{dF_1^*(a)}{da},$$

we may write

$$x = a\cos\psi + \frac{\varepsilon}{\omega}\sum_{n=2}^{\infty}\frac{nF_n^*(a)\sin n\psi}{n^2-1}, \qquad (4.7)$$

where a and ψ are defined by the equations

$$\left.\begin{aligned} \frac{da}{dt} &= \frac{\varepsilon}{2}F_1^*(a), \\ \frac{d\psi}{dt} &= \omega + \varepsilon^2 B_2(a), \end{aligned}\right\} \qquad (4.8)$$

and $B_2(a)$ has the following form:

$$B_2(a) = -\frac{1}{8a\omega}F_1^*(a)\frac{dF_1^*(a)}{da} - \frac{1}{2\omega a^2}\sum_{n=2}^{\infty}\frac{n^2F_n^{*2}(a)}{n^2-1}. \qquad (4.9)$$

On comparing the resultant approximate equations with the solutions of eq. (3.1) given in the preceding Section, we convince ourselves of their complete identity.

Thus, the system described by eq. (4.1) is likewise quasi-synchronous.

As an example, let us consider the van der Pol equation

$$\frac{d^2x}{dt^2} - \varepsilon(1 - x^2)\frac{dx}{dt} + x = 0. \qquad (4.10)$$

Comparing eqs. (4.10) and (4.1), we have

$$f(x) = 1 - x^2,$$

and, therefore,

$$F^*(x) = x - \frac{x^3}{3},$$

after which we find the expansion of (4.3) for our case

$$F^*(a\cos\psi) = a\left(1 - \frac{a^2}{4}\right)\cos\psi - \frac{a^3}{12}\cos 3\psi,$$

according to which we obtain

$$F_1^*(a) = a\left(1 - \frac{a^2}{4}\right),$$

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$$\left.\begin{aligned} F_3^*(a) &= -\frac{a^3}{12}, \\ F_n^*(a) &= 0, \quad \text{if} \quad n \neq 1, \quad n \neq 3. \end{aligned}\right\} \tag{4.11}$$

Thus, taking eq.(4.6) into consideration, we have in first approximation

$$x = a \cos \psi, \tag{4.12}$$

where a and ψ must be determined from the system of equations

$$\left.\begin{aligned} \frac{da}{dt} &= \frac{\varepsilon a}{2}\left(1 - \frac{a^2}{4}\right), \\ \frac{d\psi}{dt} &= 1. \end{aligned}\right\} \tag{4.13}$$

Thus in first approximation we obtain a harmonic oscillation having a constant frequency $\omega = 1$, whose amplitude varies according to the first differential equation of the system (4.13). To find the law of dependence of the amplitude of oscillation on the time in explicit form, this equation must be solved. On multiplying both sides of the first equation of the system (4.13) by a, we have

$$\frac{da^2}{dt} = \varepsilon\left(1 - \frac{a^2}{4}\right)a^2, \tag{4.14}$$

whence

$$\frac{da^2}{\left(1 - \frac{a^2}{4}\right)a^2} = \varepsilon\, dt,$$

or

$$\frac{da^2}{4 - a^2} + \frac{da^2}{a^2} = \varepsilon\, dt,$$

which gives

$$\ln \frac{a^2}{4 - a^2} = \ln \frac{a_0^2}{4 - a_0^2} + \varepsilon t, \tag{4.15}$$

where a_0 is the initial value of amplitude.

From eq.(4.15) we finally find

$$a = \frac{a_0 e^{\frac{1}{2}\varepsilon t}}{\sqrt{1 + \frac{1}{4}a_0^2\left(e^{\varepsilon t} - 1\right)}}. \tag{4.16}$$

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On substituting eq.(4.16) and eq.(4.12), we have an expression for the first approximation in explicit form

$$x = \frac{a_0 e^{\frac{1}{2}\varepsilon t}}{\sqrt{1 + \frac{1}{4} a_0^2 (e^{\varepsilon t} - 1)}} \cos(\omega t + \theta). \qquad (4.17)$$

As will be clear from eq.(4.17), if the initial value of the amplitude a_o is equal to zero, then the amplitude remains equal to zero for any t, and we obtain x = 0, i.e., a trivial solution of the van der Pol equation. This trivial solution obviously corresponds to the static state, i.e., to the absence of oscillations in the system.

However, starting from this same formula, it is easy to draw the conclusion that this static state is unstable. However small the initial value of the amplitude may be, it will still monotonously increase and approach a value equal to 2. In this way, since accidental small shocks are unavoidable in practice, oscillations with increasing amplitudes are automatically excited in the oscillatory system under consideration, in a state of rest, i.e., the system is self-excited.

Equation (4.17) also shows that, if $a_o = 2$, then $a = 2$ for any values of $t > 0$. This solution corresponds to the stationary (steady) dynamic state

$$x = 2\cos(t + \theta). \qquad (4.18)$$

In contrast to the static state, the dynamic state possesses excellent stability, due to the fact that, no matter whether the value of $a_o \neq 0$ is large or small, $a(t) \to 2$ will always occur while $t \to \infty$.

In other words, any oscillation with increasing t will approach the stationary oscillation of eq.(4.18).

We note that only in first approximation is it possible to represent the stationary state (4.18) as a harmonic oscillation of a frequency $\omega = 1$ and amplitude equal to 2. In reality, however, the stationary state is not harmonic.

Let us pass now to the construction of the second approximation. From equations (4.7), (4.8), and (4.11), we find

$$x = a\cos\psi - \frac{\varepsilon a^3}{32}\sin 3\psi, \qquad (4.19)$$

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where a and ψ must be determined from the equations

$$\left.\begin{aligned} \frac{da}{dt} &= \frac{\varepsilon a}{2}\left(1 - \frac{a^2}{4}\right), \\ \frac{d\psi}{dt} &= 1 - \varepsilon^2\left(\frac{1}{8} - \frac{a^2}{8} + \frac{7a^4}{256}\right). \end{aligned}\right\} \tag{4.20}$$

For stationary oscillations, in second approximation, we obtain

$$x = 2\cos(\omega t + \theta) - \frac{\varepsilon}{4}\sin 3(\omega t + \theta), \tag{4.21}$$

where

$$\omega = 1 - \frac{\varepsilon^2}{16}.$$

The above simple example of an oscillatory self-excited system, described by the van der Pol equation, shows the fundamental difference between this system and the oscillatory conservative systems described by an equation of the type of equation (2.1).

More specifically, in conservative oscillatory systems, as demonstrated above, oscillations at any constant amplitude are possible, while in self-sustained oscillatory systems oscillations at constant amplitude are possible only at a certain value of this amplitude. Physically, this is clear from the following obvious considerations. Since, in a conservative system neither dissipation nor any source of energy exist, the oscillations once excited can neither increase nor be damped, so that their amplitude will remain equal to its initial value.

In self-excited systems dissipation of energy and sources of energy exist. The amplitude of the oscillations will therefore increase if the quantity of energy delivered by the source exceeds the quantity of energy dissipated by the dissipative forces. On the other hand, if the quantity of energy supplied by the source is less than the quantity of energy dissipated, then the oscillations will be damped.

However, the amplitude will remain constant only if these quantities of energy are in exact balance.

Let us now construct approximate solutions for the van der Pol equation, utilizing the method of the mean. For this purpose, eq. (4.10) must be reduced to the standard form. This is easily done, if the unknown function x is replaced by two

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new functions a and θ, using the following formulas for the substitution of variables:

$$x = a\cos(t+\theta), \tag{4.22}$$

$$\frac{dx}{dt} = -a\sin(t+\theta). \tag{4.23}$$

Differentiating eq.(4.22) and comparing with eq.(4.23) yields

$$\frac{da}{dt}\cos(t+\theta) - a\frac{d\theta}{dt}\sin(t+\theta) = 0. \tag{4.24}$$

Differentiating eq.(4.23), and taking eqs.(4.22) and (4.10) into consideration, we get

$$\frac{da}{dt}\sin(t+\theta) + a\frac{d\theta}{dt}\cos(t+\theta) = \varepsilon[1 - a^2\cos^2(t+\theta)]\,a\sin(t+\theta). \tag{4.25}$$

On solving the eq.(4.24), (4.25) with respect to the derivatives, we arrive at the system of two equations in standard form

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon[1 - a^2\cos^2(t+\theta)]\,a\sin^2(t+\theta),\\ \frac{d\theta}{dt} &= \varepsilon[1 - a^2\cos^2(t+\theta)]\sin(t+\theta)\cos(t+\theta). \end{aligned}\right\} \tag{4.26}$$

or

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon\left\{\frac{a}{2}\left(1 - \frac{a^2}{4}\right) - \frac{a}{2}\cos 2(t+\theta) + \frac{a^3}{8}\cos 4(t+\theta)\right\},\\ \frac{d\theta}{dt} &= \varepsilon\left\{\frac{1}{2}\left(1 - \frac{a^2}{2}\right)\sin 2(t+\theta) - \frac{a^2}{8}\sin 4(t+\theta)\right\}. \end{aligned}\right\} \tag{4.27}$$

By using the method of the mean, we obtain in first approximation

$$a = a_1, \quad \theta = \theta_1,$$

where

$$\frac{da_1}{dt} = \frac{\varepsilon a_1}{2}\left(1 - \frac{a_1^2}{4}\right), \quad \frac{d\theta_1}{dt} = 0, \tag{4.28}$$

since

$$\underset{t}{M}\{\cos 2(t+\theta)\} = \underset{t}{M}\{\sin 2(t+\theta)\} = \underset{t}{M}\{\cos 4(t+\theta)\} = \underset{t}{M}\{\sin 4(t+\theta)\} = 0.$$

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Obviously, the equation of first approximation (4.28), agrees with the above-obtained equation (4.13).

The refined first approximation will obviously be

$$\left.\begin{aligned} a &= a_1 - \frac{\varepsilon a_1}{4}\sin 2(t+\theta_1) + \frac{\varepsilon a_1^3}{32}\sin 4(t+\theta_1),\\ \theta &= \theta_1 - \frac{\varepsilon}{4}\left(1 - \frac{a_1^2}{2}\right)\cos 2(t+\theta_1) + \frac{\varepsilon a_1^2}{32}\cos 4(t+\theta_1). \end{aligned}\right\} \quad (4.29)$$

For the stationary state, as above, we have

$$a(t) \to 2 \quad at \quad t \to \infty,$$

and consequently, for a steady oscillatory regime at $a_1 = 2$, the formulas (4.29) of the refined first approximation will yield

$$\left.\begin{aligned} a &= 2 - \frac{\varepsilon}{2}\sin 2(t+\theta_1) + \frac{\varepsilon}{4}\sin 4(t+\theta_1),\\ \theta &= \theta_1 + \frac{\varepsilon}{4}\cos 2(t+\theta_1) + \frac{\varepsilon}{8}\cos 4(t+\theta_1). \end{aligned}\right\} \quad (4.30)$$

On substituting these values in eq. (4.22), we get

$$\begin{aligned} x = \Big[2 - \frac{\varepsilon}{2}\sin 2(t+\theta_1) &+ \\ + \frac{\varepsilon}{4}\sin 4(t+\theta_1)\Big]\cos\Big(t+\theta_1 &+ \frac{\varepsilon}{4}\cos 2(t+\theta_1) + \frac{\varepsilon}{8}\cos 4(t+\theta_1)\Big). \end{aligned} \quad (4.31)$$

or, neglecting the terms of the second order of smallness, after elementary transformations, we obtain the refined approximation

$$x = 2\cos(t+\theta_1) - \frac{\varepsilon}{4}\sin 3(t+\theta_1), \quad (4.32)$$

which agrees with the expression for the refined approximation found earlier.

Section 5. Stationary Amplitudes and Their Stability

In the preceding Sections we have obtained approximate solutions determining the law of variation with time of the amplitudes of the fundamental oscillation harmonic.

For any n^{th} approximation, this equation will have the form

$$\frac{da}{dt} = \Phi(a), \tag{5.1}$$

where

$$\Phi(a) = \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots + \varepsilon^n A_n(a),$$

and therefore can be integrated in quadratures.

However, even without integration, it is possible to investigate the behavior of the solution $a = a(t)$ as a function of the properties of $\Phi(a)$, which will be done below:

Let us first assume that no positive quantity a^* exists for which

$$\Phi(a) > 0 \quad \text{for any} \quad a > a^*.$$

It is evidently necessary to adopt this condition for purely physical considerations.

Now, if such a quantity a^* did exist and assuming that the initial value of the amplitude a^0 is greater than a^*, it follows that

$$a(0) > a^*,$$

which would mean that, in accordance with eq.(5.1), the amplitude would increase without limit

$$a(t) \to \infty \quad as \quad t \to \infty,$$

i.e., the oscillations would broaden without limit, which is physically impossible.

For this reason, it will always be assumed in the following that this condition (which may be called the condition of limitation of amplitudes) is satisfied.

Equation (5.1) indicates that the amplitude increases when $\Phi(a) > 0$ and decreases when $\Phi(a) < 0$.

The unchanging, stationary values of a are determined by the equation

$$\Phi(a) = 0, \tag{5.2}$$

which is obtained by equating the right side of eq.(5.1) to zero.

Equation (5.1) shows that, if the initial value of the amplitude is not stationary [does not satisfy eq.(5.2)], then, with increasing time, the amplitude $a(t)$, monotonously increasing [if $\Phi(a^0) > 0$] or decreasing [if $\Phi(a^0) < 0$], will approach a stationary value.

Thus every nonstationary oscillation, with the passage of time, will approach a stationary state. Nonstationary oscillations are usually called unsteady oscillations, or oscillations in a transitional state. The fact of the approach of any oscillation to a stationary state reveals the special role of the stationary oscillations, in particular for high-frequency oscillatory processes, for which, in view of the brevity of the period of oscillation, the transient state very rapidly approaches the stationary state. For this reason, oscillations of this kind may be considered in practice as being stationary almost immediately after the beginning of the oscillatory process.

We note that there exists a case of degeneration, when the function $\Phi(a)$ is identically equal to zero. In this case there are no transient states, and every oscillation is stationary. This case occurs, for example, when $f(x, \frac{dx}{dt})$ depends only on x but not on $\frac{dx}{dt}$. Then eq.(1.1) takes the form

$$\frac{d^2x}{dt^2} + F(x) = 0, \tag{5.3}$$

which has been discussed in detail above.

This equation may be integrated as the equation of oscillations of a material point under the action of a force depending only on position and, therefore, originating in the potential

$$F(x) = \frac{dU}{dx},$$

where

$$U = \int_0^x F(x)\,dx.$$

Equation (5.3) is the equation of a conservative oscillatory system having an energy invariant throughout the oscillations.

In practice, however, no ordinary oscillatory system is conservative and always contains dissipative forces causing the dissipation of energy; likewise, a self-sustained oscillatory system may also contain energy sources.

Let us now pass to the question of the stability of stationary oscillations.

Assume that a_0 is a certain root of eq.(5.2), i.e., a constant stationary solution of eq.(5.1). Consider solutions of eq.(5.1) infinitely close to a_0. Then, using

$$a = a_0 + \delta a,$$

for an infinitely small increment δa (neglecting terms of a higher order of smallness), we obtain

$$\frac{d\,\delta a}{dt} = \Phi'(a_0)\,\delta a,$$

which gives

$$\delta a = (\delta a)_0 e^{\Phi'(a_0)t}.$$

Thus, the value of the amplitude under consideration is stable, i.e., corresponds to a stable stationary oscillation, if

$$\Phi'(a_0) < 0, \qquad (5.4)$$

Otherwise, if

$$\Phi'(a_0) > 0,$$

the corresponding stationary oscillation is obviously unstable.

In particular, since the value $a_0 = 0$, corresponding to the state of equilibrium (static state), is always a root of eq.(5.2) [by virtue of eq.(1.33)], the inequality

$$\Phi'(0) > 0$$

will represent the condition of self-excitation of the oscillations.

Writing this in the expanded form

$$\varepsilon A_1'(0) + \varepsilon^2 A_2'(0) + \ldots + \varepsilon^n A_n'(0) > 0,$$

and disregarding the case when the function $A_1(a)$ may have multiple roots, we see that, for sufficiently small values of ε (which however are always assumed either explicitly or implicitly), the problem of self-excitation is solved by the sign of a single term, namely, $\varepsilon A_1'(0)$, i.e., in the same way as though we were dealing with an equation of first approximation.

Moreover, in accordance with eq.(5.2), the stationary amplitudes must satisfy

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the equation

$$A_1(a)+\varepsilon A_2(a)+\ldots+\varepsilon^{n-1}A_n(a)=0.$$

For this reason, disregarding the above-mentioned cases of multiple roots, we can expand a into a power series of the parameter ε

$$a=a^{(0)}+\varepsilon a^{(1)}+\varepsilon^2a^{(2)}+\ldots, \tag{5.5}$$

where $a^{(0)}$ is the root of the equation $A_1(a) = 0$ (stationary amplitude in first approximation)

$$a^{(1)}=-\frac{A_2(a^{(0)})}{A_1'(a^{(0)})},\ \ldots$$

Since a given stationary state will be either stable or unstable, if respectively

$$\varepsilon A_1'(a)+\varepsilon^2A_2'(a)+\ldots+\varepsilon^nA_n'(a)\lessgtr 0,$$

and since, because of the relation

$$\varepsilon A_1'(a)+\varepsilon^2A_2'(a)+\ldots+\varepsilon^nA_n'(a)=\varepsilon A_1'(a^{(0)})+\varepsilon^2\ \ldots,$$

which results from eq.(5.5), the sign of its left side is determined (if ε is sufficiently small) by the sign of $\varepsilon A'(a^{(0)})$, we see that the problem of the stability of the stationary oscillations is solved, for sufficiently small ε, by the sign of $\varepsilon A'(a^{(0)})$, i.e., as though we were dealing with an equation of the first approximation.

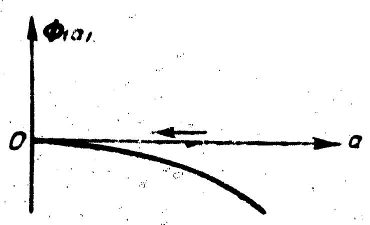

Fig. 25

It must be emphasized that, except for certain special cases, equations of the first approximation lead to the same qualitative results as those of higher approximations. The transition to equations of higher approximations usually leads only to corrections of a quantitative character, for example corrections in the value of the stationary amplitude, etc.

We note that the condition of self-excitation of the oscillation is not necessary for the existence of a stable stationary state of oscillation. For this purpose, it is obviously sufficient for the equation of the stationary state (5.2) to have at least one nonzero root, satisfying the condition (5.3).

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In addition to an analytical study of the function $\Phi(a)$, the character of the oscillatory process can, in many cases, be conveniently determined by using graphs of the type in Figs.25-28, which give $\Phi(a)$ as a function of a.

The stationary amplitudes here are determined by the points of intersection of the curve $\Phi(a)$ with the abscissa. It is obvious that the points at which the de-

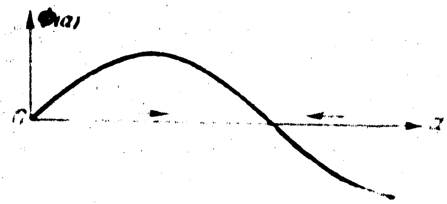

Fig.26

scending branch of the curve intersects the Oa axis correspond to a stable amplitude of the oscillations, while the points at which the ascending branch of the

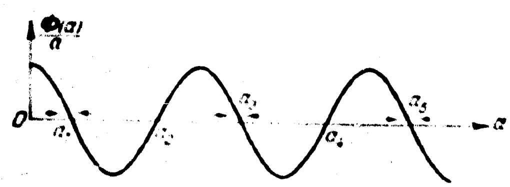

Fig.27

curve intersects the Oa axis yield unstable amplitudes.

In fact, in the former case, the impairment of the stationary amplitude leads to a subsequent variation, causing the amplitude to return to its stationary value. In the latter case we have the opposite picture. In Figs.25-28, the arrows show the direction of the variation of a. Figure 25 corresponds to the dissipative case, Fig.26 to the case of self-excitation with one possible stationary amplitude, and Fig.27 to the case of self-excitation with several stationary amplitudes: a_1, a_2, a_3 (the oscillations with amplitudes a_2, a_4, are obviously unstable).

In general, if the function $\Phi(a)$ has a root a^* satisfying the inequality

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$\Phi'(a^0) < 0$, then a stationary state of oscillations with a constant amplitude equal to a^0 is possible. We note that the stationary amplitude of self-excited oscillations (i.e., the limit of a monotonously increasing amplitude of oscillations for which a^0 would be very small) is equal to the smallest of all possible stationary amplitudes. This fact becomes clear from the logical physical consideration that an amplitude, on increasing, cannot jump over the stable root of the equation $\Phi(a) = 0$, i.e., over the root of this equation that satisfies the condition $\Phi'(a) < 0$.

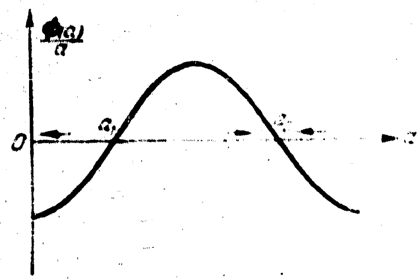

Fig. 28

Figure 28 shows the case when the system is not self-exciting, but may still contain stationary oscillations. In this case, if the initial value of the amplitudes a^0 is less than a_1, the oscillations will decay; if the initial value is greater than a_1, the oscillations will build up and, at the limit, will be transformed into stationary oscillations of an amplitude a_2.

As an example, let us consider the equation

$$\frac{d^2x}{dt^2} + (\lambda_1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3 + \lambda_5 x^4)\frac{dx}{dt} + \omega^2 x = 0, \tag{5.6}$$

which is encountered in the theory of vacuum-tube oscillators.

To assure the applicability of the results obtained by us from this equation, let us assume that the perturbation term is sufficiently small, and let us put

$$-(\lambda_1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3 + \lambda_5 x^4)\frac{dx}{dt} = \varepsilon f\left(x, \frac{dx}{dt}\right).$$

Then we get

$$\frac{\varepsilon}{\pi}\int_0^{2\pi} f(a\cos\psi, \; -a\omega\sin\psi)\sin\psi\, d\psi = \lambda_1 a\omega + \frac{\lambda_3 a^3\omega}{4} + \frac{\lambda_5 a^5\omega}{8}$$

Consequently, eq.(1.27) will yield the following equation of first approximation

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for the amplitude

$$\frac{da}{dt} = -\frac{\lambda_1 a}{2} - \frac{\lambda_3 a^3}{8} - \frac{\lambda_5 a^5}{16}.$$

We note that, if $\lambda_5 < 0$, the right-hand side of this equation is positive for all sufficiently large values of a. Thus, in this case, oscillations with a sufficiently great amplitude will broaden without limit, i.e., $a(t) \to \infty$ as $t \to \infty$, which is obviously impossible for physical reasons.

Let us, therefore, assume that $\lambda_5 > 0$. We further note that the condition of self-excitation will be $\lambda_1 < 0$. On considering the case of the absence of self-excitation, we put $\lambda_1 > 0$. On solving the equation

$$\frac{\lambda_1 a}{2} + \frac{\lambda_3 a^3}{8} + \frac{\lambda_5 a^5}{16} = 0, \tag{5.7}$$

in addition to the "static" solution $a = 0$, we also find

$$a^2 = -\frac{\lambda_3}{\lambda_5} \pm \sqrt{\left(\frac{\lambda_3}{\lambda_5}\right)^2 - \frac{8\lambda_1}{\lambda_5}}.$$

Since $\lambda_1 > 0$, $\lambda_2 > 0$, then, if $\lambda_3 > 0$ or if $\left(\frac{\lambda_3}{\lambda_5}\right)^2 < \frac{8\lambda_1}{\lambda_5}$, eq.(5.7) has no positive solutions. The graph obtained by plotting $\frac{da}{dt}$ against a will have the form shown in Fig.25, indicating that oscillations of any amplitude will be damped.

Let, on the other hand,

$$\lambda_3 < 0, \quad \left(\frac{\lambda_3}{\lambda_5}\right)^2 > \frac{8\lambda_1}{\lambda_5};$$

Then we have two possible solutions for the amplitude of the stationary oscillations

$$a_1 = \sqrt{-\frac{\lambda_3}{\lambda_5} - \sqrt{\left(\frac{\lambda_3}{\lambda_5}\right)^2 - \frac{8\lambda_1}{\lambda_5}}},$$

$$a_2 = \sqrt{-\frac{\lambda_3}{\lambda_5} + \sqrt{\left(\frac{\lambda_3}{\lambda_5}\right)^2 - \frac{8\lambda_1}{\lambda_5}}}.$$

The graph for $\frac{da}{dt}$ versus a, as obtained for this case, is presented in Fig.29. Obviously a_1 corresponds to unstable oscillations and a_2 to stable types.

Thus, oscillations with an initial amplitude smaller than a_1 will be damped,

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while oscillations with an initial amplitude greater than a_1, will approach a stable stationary state.

Let us now consider the case when an oscillatory system contains a certain parameter μ (or a certain group of parameters) which may vary as slowly as desired (adiabatically). In this case, the right side of eq.(5.1) will depend on μ and may be represented in the form $\Phi(a,\mu)$.

We will consider a variation of the parameter that is so slow by comparison

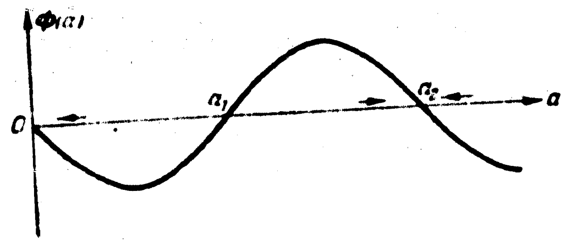

Fig.29

with the effective duration of the transient state (i.e., by comparison with the time during which an arbitrary oscillation becomes practically stationary), that for each value of μ during the course of this variation, the oscillation may be assumed to be stationary.

For the sake of definiteness we assume that for values of μ smaller than a certain μ_e,

$$\Phi'_a(0, \mu) < 0,$$

and for values of μ larger than these values,

$$\Phi'_a(0, \mu) > 0.$$

We shall now vary μ adiabatically, increasing it from a certain value μ_1, less than μ_e.

Let the system be initially in equilibrium: $a = 0$. Then, since for $\mu < \mu_e$, the system is not self-exciting, it will also remain in equilibrium until the parameter μ of the critical value equal to μ_e is reached. For the transition through this

critical value, self-excitation will appear, while equilibrium becomes impossible, and the amplitude a passes from zero to the value $a(\mu)$ equal to the smallest stable root of the equation

$$\Phi(a, \mu) = 0.$$

Thus the dependence of the amplitude on the parameter is represented in the form

$$a = 0 \quad \text{for} \quad \mu < \mu_0,$$
$$a = a(\mu) \quad \text{for} \quad \mu > \mu_0.$$

If the curve of the dependence of a and μ so obtained is continuous, we will say that we have a case of soft excitation of oscillations (with respect to the given parameter). In the case of soft excitation, on transition across the critical value $\mu = \mu_0$, the system will begin to generate oscillations whose amplitude, close to the critical value, will gradually increase from zero.

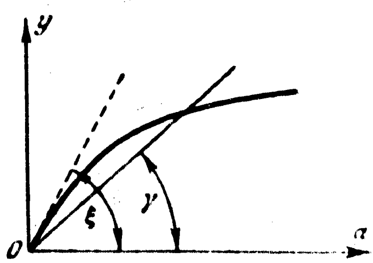

Fig. 30

However, if there is a discontinuity at the point $\mu = \mu_0$, then, on transition through the critical value, the amplitude will immediately jump from the zero value to the value $a(\mu_0 + 0)$. This case is called the case of hard excitation.

For example, let the right side of eq.(5.1) have the following form:

$$\Phi(a, \mu) = \left\{ \Phi(a) - \frac{a}{\mu} \right\} \Psi(a, \mu), \qquad (5.8)$$

where $\Psi(a,\mu) > 0$, and where $\Phi(a)$ is a certain function, not dependent on μ.

In this case, the question of the character of the excitation may be decided by the aid of one of the two following graphic constructions:

Let us construct the curve (cf. Fig. 30)

$$y = \Phi(a).$$

Then the stationary amplitudes will be found from the intersections of this

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curve with the straight lines of the form $y = \frac{1}{\mu}a$. If the slope of the straight line is greater than the slope of the tangent to the point of intersection, the stationary amplitude will be stable.

In Fig.30 we have a case of soft excitation since, as soon as μ passes through the critical value, equal to

$$\mu_0 = \frac{1}{\operatorname{tg}\xi} = \frac{1}{\Phi'_a(0)},$$

the amplitude begins to increase from zero.

In Fig.31 we have a case of hard excitation. In transition through the critical value, the amplitude jumps from zero to a_1.

Let us consider in detail the case represented in Fig.31. Let the parameter μ

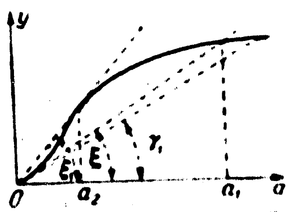

Fig.31

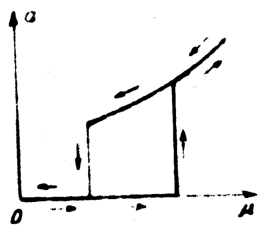

Fig.32

be gradually increased from zero, and the angle γ of the slope of the straight line $y = \frac{1}{\mu}a$ be decreased in the same way, and let the system be initially in the state of rest $a = 0$.

Then obviously the amplitude will remain zero until the moment when γ becomes less than ξ. After the transition through ξ, the amplitude will jump to the value a_1 and will then begin to increase continuously. If we now begin to decrease μ (increase γ), beginning from the value $\gamma_1 > \xi$, then the amplitude will decrease and, beginning with $\gamma = \xi_1$, will break off; in this case, a state of rest will be established in the system. When plotting the dependence of a on μ in the course of such process, we obtain the curves (Fig.32), having a characteristic hysteresis loop. The value of the stationary amplitude will depend not only on the value of the parameter, but also the preceding variation of this parameter.

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Phenomena of similar type are observed in certain self-sustained oscillatory systems and are called oscillatory hysteresis or lag; the latter term reflects the fact that, at an infinitely slow or, as it is sometimes called, an adiabatic variation of the parameter, the amplitude tends, as it were, to drag out its smooth variation for as long as possible until the continuous variation leads to unstable amplitudes.

Hysteresis loops may also have a more complex form than that shown in Fig.32. For example, for the case of the diagram in Fig.33, a relationship as schematically

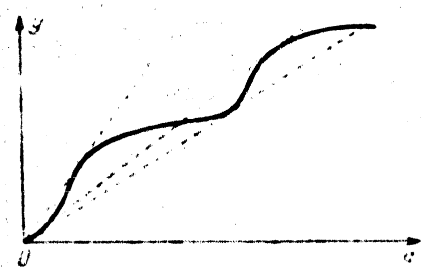

Fig.33

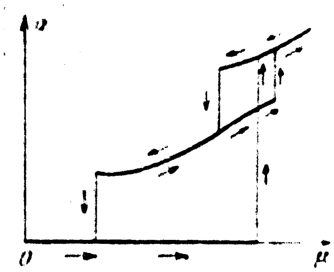

Fig.34

shown in Fig.34 is obtained on variations in μ. To study the character of the excitation of the oscillations, the following method which is essentially analogous to the above-described method, may also be used.

Let us construct the curve

$$y = \frac{\Phi(a)}{a}. \tag{5.9}$$

Then the stationary amplitudes will be found at the intersections of this curve with the straight lines parallel to the a axis

$$y = \frac{1}{\mu}. \tag{5.10}$$

The condition of stability

$$\frac{\partial \Phi(a, \mu)}{\partial a} < 0$$

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may be written in the form

$$\frac{\partial\left\{\frac{\Phi(a)}{a}-\frac{1}{\mu}\right\}}{\partial a}<0 \tag{5.11}$$

and therefore allows a simple geometric interpretation. Amplitudes corresponding to intersection points of the curve of eq.(5.9) with a straight line of eq.(5.10), at which the curve has an ascending direction, will be stable (Fig.35).

These statements refer to stationary amplitudes not equal to zero. In addition, there always exist stationary amplitudes which are equal to zero. They will be unstable if

$$\left\{\frac{\Phi(a)}{a}\right\}_{a=0}>\frac{1}{\mu}, \tag{5.12}$$

i.e., if the point of intersection of the curve (5.9) with the axis a = 0 lies above the point of intersection of the straight line of eq.(5.10) with this curve. In the opposite case, the equilibrium is stable.

It is clear that the latter graphic method is more convenient than the former

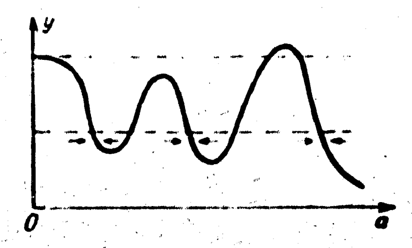

Fig.35

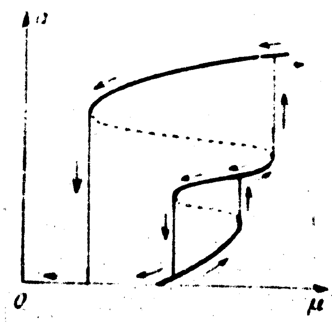

Fig.36

one, since in this case the tangents need not be drawn. Otherwise the reasoning remains the same; therefore, a detailed discussion may be omitted.

Figure 36 schematically shows the pattern of oscillatory hysteresis corresponding to the case represented by Fig.35.

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As an extremely simple example, let us consider the equation of the electronic oscillator (with dimensionless time):

$$\frac{d^2V}{dt^2} + V + \frac{1}{\sqrt{LC}}\left\{\frac{L}{R} - (M - DL) f'(E_0 + V)\right\}\frac{dV}{dt} = 0. \tag{5.13}$$

Here V is the alternating component of the control voltage; L, C, and R are inductance, capacitance and resistance of the oscillatory circuit, respectively; M is the coefficient of mutual inductance between the grid circuit and the oscillatory circuit; E_0 is a constant component of the grid voltage; D is the grid through of the tube.

In view of the fact that, at a conventional design of the oscillator, the dimensionless expression

$$\frac{1}{\sqrt{LC}}\left\{\frac{1}{R} - (M - DL) f'(E_0 + V)\right\} \tag{5.14}$$

will be a quantity of the order of 0.01, we may utilize the formulas derived previously for the construction of an approximate solution here.

On the basis of eqs.(1.23), (1.24) and (1.27), in first approximation, we have

$$V = a\cos(t + \varphi), \tag{5.15}$$

while

$$2\sqrt{LC}\frac{da}{dt} = -\frac{L}{R}a + (M - DL)F(a), \tag{5.16}$$

where

$$F(a) = \frac{a}{\pi}\int_0^{2\pi} f'(E_0 + a\cos\tau)\sin^2\tau\, d\tau = -\frac{1}{\pi}\int_0^{2\pi} f(E_0 + a\cos\tau)\sin\tau\, d\tau. \tag{5.17}$$

According to the above, the character of the excitation of oscillations as expressed by eq.(5.16), according to the above, may be studied by two graphic methods:

1) By constructing the curve

$$y = F(a),$$

of the so-called oscillatory characteristic of the tube. In this case, the stationary amplitude will be found from the intersection of this curve with the straight

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lines

$$y = \frac{L}{(M - DL)R}\,a.$$

2) By constructing the curve

$$y = \frac{F(a)}{a}$$

and considering its intersections with the straight lines

$$y = \frac{L}{(M - DL)R},$$

parallel to the axis $y = 0$.

It should be mentioned that the angular coefficient of the tangent to the tube characteristic, i.e., $f'(E)$ is usually called the grid-plate transconductance in radio engineering.

In view of the fact that

$$\frac{F(a)}{a} = \frac{1}{\pi}\int_0^{2\pi} f'(E_0 + a\cos\tau)\sin^2\tau\, d\tau = f'(E_0 + a\cos\vartheta)$$
$$(0 \leqslant \vartheta \leqslant 2\pi),$$

the expression $\frac{F(a)}{a}$ may be called the mean transconductance of the tube.

For this reason, the first graphic method is sometimes called the method of the oscillatory characteristic, and the second, the method of the mean transconductance.

In this example, the parameter $\mu = \frac{(M - DL)R}{L}$.

In the present Section we have considered in detail the first equation of the system (1.3), expressing the dependence of the amplitude on the time and thereby characterizing the properties of the oscillatory process with respect to its amplitude.

As for the second equation of the system (1.3), it characterizes the frequency properties of the oscillations.

According to this equation, the instantaneous natural frequency of the oscillations, $\frac{d\psi}{dt}$ equals $\omega(a)$. For this reason, in the case of stationary oscillations, $\omega(a)$, being a constant, will be the ordinary natural frequency.

It is easy to see that the natural frequency $\omega(a)$, and thus also the period

$T = \frac{2\pi}{\omega(a)}$ of the stationary oscillations, depends on the amplitude. Thus, speaking generally, nonlinear oscillatory systems are not isochronous.

As we have seen above, in some important special cases, a nonlinear oscillatory system may be isochronous in first approximation.

Section 6. Construction of Stationary Solutions

In the preceding Sections we have presented a method of constructing approximate solutions for equations of the type of eq.(1.1). The first and higher approximations have been constructed for various special cases of eq.(1.1) and calculations for specific examples have also been made. As proved there, the solution of a nonlinear differential equation of the type of eq.(1.1) is in all cases replaced by the solution of two first-order equations defining the amplitude and phase of the oscillation.

Thus, in order to construct an approximate solution with a definite and predetermined degree of accuracy, we must set up equations of the type of eq.(1.5) and must then use them for determining expressions for the amplitude and phase, as a function of time.

To determine the approximate solutions, corresponding to a steady state in an oscillatory system (stationary oscillations), the right side of eq.(5.1) must be equated to zero since, in a stationary state, the amplitude is constant, and consequently, its derivative is zero. The resultant algebraic equation gives the stationary values of the amplitude. However, for constructing the approximate solutions, corresponding directly to the stationary oscillations, a simpler method than the above process can be given:

Let us first consider the equation of the conservative oscillatory system of eq.(2.1), which may be written in the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f(x). \qquad (6.1)$$

According to the results of Section 2, the stationary solution of this equation in second approximation has the form

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$$\left.\begin{aligned} x &= a\cos(\omega t+\varphi) - \frac{\varepsilon}{\omega^2}\sum_{\substack{n=0\\ n\neq 1}}^{\infty}\frac{f_n(a)\cos n(\omega t+\varphi)}{n^2-1},\\ \omega_{II}^2(a) &= \omega^2 - \varepsilon\frac{f_1(a)}{a} + \varepsilon^2\ldots, \end{aligned}\right\} \qquad (6.2)$$

where $f_n(a)$ ($n = 0, 1, 2, \ldots$) are the Fourier coefficients in the expansions

$$\left.\begin{aligned} f(a\cos\psi) &= \sum_{n=0}^{\infty} f_n(a)\cos n\psi,\\ f_n(a) &= \frac{1}{\pi}\int_0^{2\pi} f(a\cos\psi)\cos n\psi\, d\psi, \end{aligned}\right\} \qquad (6.3)$$

while a and φ are the integration constants determined by the initial values.

Starting from eq.(6.2), it is natural to make use of the following method to obtain the higher approximations corresponding to the stationary state:

Let us represent the solution of eq.(6.1) in the form $x = z(\bar{\omega}t + \varphi)$, where $z(\bar{\omega}t + \varphi)$ is a periodic function of $\bar{\omega}t + \varphi$, with the period 2π.

We note that $x = z(\bar{\omega}t + \varphi)$ will satisfy eq.(6.1) only when $z(\bar{\omega}t + \varphi)$ satisfies the equation

$$\bar{\omega}^2\frac{d^2 z}{d\psi^2} + \omega^2 z = \varepsilon f(z). \qquad (6.4)$$

The solution of eq.(6.4), $z = z(\psi)$, $\psi = \bar{\omega}t + \varphi$, and also the expression for the frequency of the oscillation $\bar{\omega}$ will naturally be sought in the form of the expansions

$$z(\psi) = \sum_{n=0}^{\infty}\varepsilon^n z_n(\psi), \qquad (6.5)$$

$$\bar{\omega}^2 = \sum_{n=0}^{\infty}\varepsilon^n \alpha_n, \qquad (6.6)$$

whose coefficients are determined by substituting eqs.(6.5) and (6.6) in eq.(6.4) and equating the coefficients for similar powers of the small parameter ε; in this case, we require that the $z_n(\psi)$ are periodic functions of ψ, with the period 2π.

By performing this substitution, we obtain the following equations:

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$$\left.\begin{aligned}
&\alpha_0 \frac{d^2 z_0}{d\psi^2} + \omega^2 z_0 = 0,\\
&\alpha_0 \frac{d^2 z_1}{d\psi^2} + \omega^2 z_1 = f(z_0) - \alpha_1 \frac{d^2 z_0}{d\psi^2},\\
&\alpha_0 \frac{d^2 z_2}{d\psi^2} + \omega^2 z_2 = f'(z_0) z_1 - \alpha_2 \frac{d^2 z_0}{d\psi^2} - \alpha_1 \frac{d^2 z_1}{d\psi^2},\\
&\alpha_0 \frac{d^2 z_3}{d\psi^2} + \omega^2 z_3 = f'(z_0) z_2 + \frac{1}{2} f''(z_0) z_1^2 - \alpha_3 \frac{d^2 z_0}{d\psi^2} -\\
&\qquad\qquad - \alpha_2 \frac{d^2 z_1}{d\psi^2} - \alpha_1 \frac{d^2 z_2}{d\psi^2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{6.7}$$

On determining, from the first $N + 1$ equations of the system (6.7), the functions $z_0, z_1, z_2, \ldots, z_N$, and also the quantities $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_N$, we may set up the expression

$$x = \sum_{n=0}^{N} \varepsilon^n z_n (\bar{\omega} t + \varphi), \tag{6.8}$$

where

$$\bar{\omega}^2 = \sum_{n=0}^{N} \varepsilon^n \alpha_n,$$

which will satisfy eq.(6.1) with an accuracy to quantities of the order of smallness of ε^{N+1} and, consequently, may be regarded as the $(N + 1)^{th}$ approximation of the solution of eq.(6.1), corresponding to stationary oscillations. The functions $z_n(\bar{\omega} t + \varphi)$ and the quantities $\alpha_n (n = 0, 1, 2, \ldots)$, from eq.(6.7), cannot be determined uniquely. In order to determine these quantities uniquely, it is necessary to impose certain additional conditions.

Let us require that $z_n(\psi)$ $(n = 1, 2, 3, \ldots)$ must not contain the fundamental harmonic of the argument ψ. From the first equation of (6.7) we find

$$z_0(\psi) = a \cos \psi, \quad \alpha_0 = \omega^2. \tag{6.9}$$

On substituting eq.(6.9) in the second equation of the system (6.7), we find

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$$\omega^2\left(\frac{d^2z_1}{d\psi^2}+z_1\right)=f(a\cos\psi)+\alpha_1 a\cos\psi, \tag{6.10}$$

or, taking eq.(6.3) into consideration:

$$\omega^2\left(\frac{d^2z_1}{d\psi^2}+z_1\right)=\sum_{\substack{n=0\\ n\neq 1}}^{\infty} f_n(a)\cos n\psi+\left(\alpha_1 a+f_1(a)\right)\cos\psi. \tag{6.11}$$

Bearing in mind the requirement that the function $z_1(\psi)$ is to be periodic let us equate to zero the coefficient of the first harmonic of the argument ψ in the right side of eq.(6.11). As a result, we obtain the following equation for determining α_1:

$$\alpha_1 a+f_1(a)=0,$$

whence we find

$$\alpha_1=\frac{-f_1(a)}{a}.$$

On substituting the resultant value of α_1 in the right side of eq.(6.11), we have

$$\frac{d^2z_1}{d\psi^2}+z_1=\frac{1}{\omega^2}\sum_{\substack{n=0\\ n\neq 1}}^{\infty} f_n(a)\cos n\psi, \tag{6.12}$$

whence we find the following expression for $z_1(\psi)$:

$$z_1(\psi)=\frac{1}{\omega^2}\sum_{\substack{n=\\ n\neq 1}}^{\infty}\frac{f_n(a)\cos n\psi}{1-n^2}. \tag{6.13}$$

In this case, both eq.(6.13) and the expression for α_1 coincide with the expressions found by the general method.

By continuing this process we may successively determine all the functions z_1, z_2, z_3, ... and the quantities α_1, α_2, α_3, ... to any desired subscript, no matter how large, and may thus construct approximate solutions satisfying eq.(6.1), with an accuracy to any desired power of ε.

As an example, let us determine the stationary solution in the third refined approximation (with an accuracy to quantities of the order of smallness of ε^3 in-

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clusive), for the oscillatory system described by an equation of the form

$$\frac{d^2x}{dt^2} + x + \varepsilon x^3 = 0. \qquad (6.14)$$

For determining the functions z_0, z_1, z_2, z_3 and the quantities α_0, α_1, α_2, α_3, on the basis of eq.(6.7), we obtain the equations

$$\left.\begin{aligned} &\alpha_0 \frac{d^2z_0}{d\psi^2} + z_0 = 0, \\ &\alpha_0 \frac{d^2z_1}{d\psi^2} + z_1 = -z_0^3 - \alpha_1 \frac{d^2z_0}{d\psi^2}, \\ &\alpha_0 \frac{d^2z_2}{d\psi^2} + z_2 = -3z_0^2z_1 - \alpha_2 \frac{d^2z_0}{d\psi^2} - \alpha_1 \frac{d^2z_1}{d\psi^2}, \\ &\alpha_0 \frac{d^2z_3}{d\psi^2} + z_3 = -3z_0^2z_2 - 3z_1z_0 - \alpha_3 \frac{d^2z_0}{d\psi^2} - \\ &\qquad\qquad - \alpha_2 \frac{d^2z_1}{d\psi^2} - \alpha_1 \frac{d^2z_2}{d\psi^2}. \end{aligned}\right\} \qquad (6.15)$$

From the first equation we find

$$z_0(\psi) = a\cos\psi, \quad \alpha_0 = 1. \qquad (6.16)$$

After this, the second equation may be written in the form

$$\frac{d^2z_1}{d\psi^2} + z_1 = -a^3\cos^3\psi + \alpha_1 a\cos\psi, \qquad (6.17)$$

or

$$\frac{d^2z_1}{d\psi^2} + z_1 = -\frac{a^3}{4}\cos 3\psi + \left(\alpha_1 a - \frac{3}{4}a^3\right)\cos\psi, \qquad (6.18)$$

whence we have

$$\left.\begin{aligned} \alpha_1 &= \frac{3}{4}a^2, \\ z_1(\psi) &= \frac{a^3}{32}\cos 3\psi. \end{aligned}\right\} \qquad (6.19)$$

On substituting eqs.(6.16) and (6.19) in the third equation of the system (6.15), we get

$$\frac{d^2z_2}{d\psi^2} + z_2 = -\frac{3a^5}{32}\cos^2\psi\cos 3\psi + \frac{27}{128}a^5\cos 3\psi + \alpha_2 a\cos\psi, \qquad (6.20)$$

or

$$\frac{d^2z_2}{d\psi^2}+z_2=\frac{21}{128}a^5\cos 3\psi-\frac{3a^5}{128}\cos 5\psi+\left(\alpha_2 a-\frac{3a^5}{128}\right)\cos\psi, \quad (6.21)$$

whence we find

$$\left.\begin{aligned}\alpha_2&=\frac{3a^4}{128},\\ z_2(\psi)&=-\frac{21}{1024}a^5\cos 3\psi+\frac{a^5}{1024}\cos 5\psi.\end{aligned}\right\} \quad (6.22)$$

After this, the last equation of the system (6.15) may be written in the form

$$\begin{aligned}\frac{d^2z_3}{d\psi^2}+z_3=&-3a^2\cos^2\psi\left(-\frac{21}{1024}a^5\cos 3\psi+\frac{a^5}{1024}\cos 5\psi\right)-\\ &-3\frac{a^6}{1024}\cos^2 3\psi\cdot a\cos\psi+\alpha_3 a\cos\psi+\frac{3a^4}{128}\frac{9a^3}{32}\cos 3\psi-\\ &-\frac{3}{4}a^2\left(\frac{21}{1024}a^5\,9\cos 3\psi-\frac{a^5}{1024}\,25\cos 5\psi\right),\end{aligned} \quad (6.23)$$

or

$$\begin{aligned}\frac{d^2z_3}{d\psi^2}+z_3=&-\frac{1059a^7}{2048}\cos 3\psi+\frac{177}{2048}a^7\cos 5\psi-\\ &-\frac{3}{2048}a^7\cos 7\psi+\left(\alpha_3 a+\frac{57}{4096}a^7\right)\cos\psi,\end{aligned} \quad (6.24)$$

from which we find

$$\left.\begin{aligned}\alpha_3&=-\frac{57}{4096}a^6,\\ z_3(\psi)&=\frac{1059}{2048\cdot 8}a^7\cos 3\psi-\frac{177}{2048\cdot 24}a^7\cos 5\psi+\frac{3}{2048\cdot 48}a^7\cos 7\psi.\end{aligned}\right\} \quad (6.25)$$

Thus, taking eqs.(6.8), (6.19), (6.22), and (6.25) into consideration, we obtain the approximate stationary solution of eq.(6.14) with an accuracy to quantities of the order of smallness of ε^3 inclusive, in the form

$$\begin{aligned}x=&\,a\cos(\bar\omega t+\varphi)+\varepsilon\frac{a^3}{32}\left[1-\varepsilon\frac{21}{32}a^2+\varepsilon^2\frac{1059}{512}a^4\right]\cos 3(\bar\omega t+\varphi)+\\ &+\varepsilon^2\frac{a^5}{1024}\left[1-\varepsilon\frac{177}{48}a^2\right]\cos 5(\bar\omega t+\varphi)+\varepsilon^3\frac{3a^7}{2048\cdot 48}\cos 7(\bar\omega t+\varphi),\end{aligned} \quad (6.26)$$

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where a and φ are arbitrary constants, and the frequency of the oscillations is determined by the expression

$$\bar{\omega}^2 = 1 + \frac{3}{4}\varepsilon a^2 + \frac{3}{128}\varepsilon^2 a^4 - \frac{57}{4096}\varepsilon^3 a^6. \qquad (6.27)$$

Let us pass now to the construction of approximate solutions for stationary oscillations in nonconservative systems. For this purpose, let us consider the equation of the form of eq.(1.1)

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}\right). \qquad (6.28)$$

The stationary solution (refined first approximation) of this equation, according to Section 2, may be written in the form

$$x = a\cos\psi + \frac{g_0(a)}{\omega^2} - \frac{1}{\omega^2}\sum_{n=2}^{\infty}\frac{g_n(a)\cos n\psi + h_n(a)\sin n\psi}{n^2 - 1}, \qquad (6.29)$$

where $g_n(a)$ and $h_n(a)$ ($n = 0, 2, 3, \ldots$) are determined by eq.(1.29), while $\psi = \bar{\omega}(a)t + \varphi$, and a and $\bar{\omega}(a)$ must be found from the following expressions:

$$\left.\begin{aligned} A_1(a) &= 0, \\ \bar{\omega}(a) &= \omega + \varepsilon B_1(a), \end{aligned}\right\} \qquad (6.30)$$

and $A_1(a)$ and $B_1(a)$ are determined from eq.(1.27).

For the case of conservative oscillatory systems, as proved above, $A_1(a)$ vanishes identically; for this reason, the expression for the approximate stationary solution eq.(6.29) depends on the two arbitrary constants a and φ.

Let us now consider the case when $A_1(a)$ does not identically vanish in any interval of values of a. Let us also assume that the function $A_1(a)$ has only simple roots. In this case, to each root of $A_1(a)$ will correspond a certain stationary state and eq.(6.29), for this stationary state, will depend only on a single arbitrary constant, namely on φ.

On proceeding to a discussion of the formal technique of constructing the stationary solutions of eq.(1.1), we are mainly going to use the above methods for conservative oscillatory systems.

The solution of eq.(6.28), corresponding to a stationary oscillation will be

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presented in the form

$$x = z(\bar{\omega}t + \varphi), \tag{6.31}$$

where φ is an arbitrary constant, $\bar{\omega}$ the frequency of oscillation; $z(\psi)$ a periodic function of ψ, with the period 2π.

As above, the function $z(\psi)$ must satisfy the equation

$$\bar{\omega}^2 \frac{d^2 z}{d\psi^2} + \omega^2 z = \varepsilon f\left(z,\ \frac{dz}{d\psi}\right). \tag{6.32}$$

We will determine the function $z(\psi)$ and the frequency of oscillations $\bar{\omega}$ in the form of the expansions

$$\left.\begin{aligned} z(\psi) &= \sum_{n=0}^{\infty} \varepsilon^n z_n(\psi), \\ \bar{\omega} &= \sum_{n=0}^{\infty} \varepsilon^n \omega_n, \end{aligned}\right\} \tag{6.33}$$

where $z_n(\psi)$ are all periodic functions of ψ, with the period 2π.

On substituting the values of $z(\psi)$ and $\bar{\omega}$ from eq.(6.33) in eq.(6.32), and equating the coefficients of equal powers of ε, we obtain the following system of equations:

$$\left.\begin{aligned} &\omega_0^2 \frac{d^2 z_0}{d\psi^2} + \omega^2 z_0 = 0, \\ &\omega_0^2 \frac{d^2 z_1}{d\psi^2} + \omega^2 z_1 = f\left(z_0,\ \omega_0 \frac{dz_0}{d\psi}\right) - 2\omega_0\omega_1 \frac{d^2 z_0}{d\psi^2}, \\ &\omega_0^2 \frac{d^2 z_2}{d\psi^2} + \omega^2 z_2 = f'_z\left(z_0,\ \omega_0 \frac{dz_0}{d\psi}\right) z_1 + \\ &\quad + f'_{z'}\left(z_0,\ \omega_0 \frac{dz_0}{d\psi}\right)\omega_0 \frac{dz_1}{d\psi} + f'_{z'}\left(z_0,\ \omega_0 \frac{dz_0}{d\psi}\right)\omega_1 \frac{dz_0}{d\psi} - \\ &\quad - 2\omega_0\omega_2 \frac{d^2 z_0}{d\psi^2} - 2\omega_0\omega_1 \frac{d^2 z_1}{d\psi^2} - \omega_1^2 \frac{d^2 z_0}{d\psi^2}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{6.34}$$

Let us solve the first of these equations, putting:

$$z_0 = a\cos\psi, \quad \omega_0 = \omega, \tag{6.35}$$

where a is a certain still undetermined constant.

On substituting the values of $z(\psi)$ and ω_0 from eq.(6.35) in the right side of

where a and φ are arbitrary constants, and the frequency of the oscillations is determined by the expression

$$\bar{\omega}^2 = 1 + \frac{3}{4}\varepsilon a^2 + \frac{3}{128}\varepsilon^2 a^4 - \frac{57}{4096}\varepsilon^3 a^6. \qquad (6.27)$$

Let us pass now to the construction of approximate solutions for stationary oscillations in nonconservative systems. For this purpose, let us consider the equation of the form of eq.(1.1)

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}\right). \qquad (6.28)$$

The stationary solution (refined first approximation) of this equation, according to Section 2, may be written in the form

$$x = a\cos\psi + \frac{g_0(a)}{\omega^2} - \frac{1}{\omega^2}\sum_{n=2}^{\infty}\frac{g_n(a)\cos n\psi + h_n(a)\sin n\psi}{n^2 - 1}, \qquad (6.29)$$

where $g_n(a)$ and $h_n(a)$ ($n = 0, 2, 3, \ldots$) are determined by eq.(1.29), while $\psi = \bar{\omega}(a)t + \varphi$, and a and $\bar{\omega}(a)$ must be found from the following expressions:

$$\left.\begin{aligned} A_1(a) &= 0, \\ \bar{\omega}(a) &= \omega + \varepsilon B_1(a), \end{aligned}\right\} \qquad (6.30)$$

and $A_1(a)$ and $B_1(a)$ are determined from eq.(1.27).

For the case of conservative oscillatory systems, as proved above, $A_1(a)$ vanishes identically; for this reason, the expression for the approximate stationary solution eq.(6.29) depends on the two arbitrary constants a and φ.

Let us now consider the case when $A_1(a)$ does not identically vanish in any interval of values of a. Let us also assume that the function $A_1(a)$ has only simple roots. In this case, to each root of $A_1(a)$ will correspond a certain stationary state and eq.(6.29), for this stationary state, will depend only on a single arbitrary constant, namely on φ.

On proceeding to a discussion of the formal technique of constructing the stationary solutions of eq.(1.1), we are mainly going to use the above methods for conservative oscillatory systems.

The solution of eq.(6.28), corresponding to a stationary oscillation will be

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presented in the form

$$x = z(\bar{\omega}t + \psi), \tag{6.31}$$

where ψ is an arbitrary constant, $\bar{\omega}$ the frequency of oscillation; $z(\psi)$ a periodic function of ψ, with the period 2π.

As above, the function $z(\psi)$ must satisfy the equation

$$\bar{\omega}^2 \frac{d^2 z}{d\psi^2} + \omega^2 z = \varepsilon f\left(z, \frac{dz}{d\psi}\right). \tag{6.32}$$

We will determine the function $z(\psi)$ and the frequency of oscillations $\bar{\omega}$ in the form of the expansions

$$\left.\begin{aligned} z(\psi) &= \sum_{n=0}^{\infty} \varepsilon^n z_n(\psi), \\ \bar{\omega} &= \sum_{n=0}^{\infty} \varepsilon^n \omega_n, \end{aligned}\right\} \tag{6.33}$$

where $z_n(\psi)$ are all periodic functions of ψ, with the period 2π.

On substituting the values of $z(\psi)$ and $\bar{\omega}$ from eq.(6.33) in eq.(6.32), and equating the coefficients of equal powers of ε, we obtain the following system of equations:

$$\left.\begin{aligned} &\omega_0^2 \frac{d^2 z_0}{d\psi^2} + \omega^2 z_0 = 0, \\ &\omega_0^2 \frac{d^2 z_1}{d\psi^2} + \omega^2 z_1 = f\left(z_0, \omega_0 \frac{dz_0}{d\psi}\right) - 2\omega_0\omega_1 \frac{d^2 z_0}{d\psi^2}, \\ &\omega_0^2 \frac{d^2 z_2}{d\psi^2} + \omega^2 z_2 = f'_z\left(z_0, \omega_0 \frac{dz_0}{d\psi}\right) z_1 + \\ &\quad + f'_{z'}\left(z_0, \omega_0 \frac{dz_0}{d\psi}\right)\omega_0 \frac{dz_1}{d\psi} + f'_{z'}\left(z_0, \omega_0 \frac{dz_0}{d\psi}\right)\omega_1 \frac{dz_0}{d\psi} - \\ &\quad - 2\omega_0\omega_2 \frac{d^2 z_0}{d\psi^2} - 2\omega_0\omega_1 \frac{d^2 z_1}{d\psi^2} - \omega_1^2 \frac{d^2 z_0}{d\psi^2}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{6.34}$$

Let us solve the first of these equations, putting:

$$z_0 = a \cos\psi, \quad \omega_0 = \omega, \tag{6.35}$$

where a is a certain still undetermined constant.

On substituting the values of $z(\psi)$ and ω_0 from eq.(6.35) in the right side of

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the second equation of the system (6.34) and taking eq.(1.16) into consideration, we have

$$\omega^2\left(\frac{d^2z_1}{d\psi^2}+z_1\right)=\sum_{n=0}^{\infty}[g_n(a)\cos n\psi+h_n(a)\sin n\psi]+2\omega\omega_1 a\cos\psi. \quad (6.36)$$

To give this equation a periodic solution with respect to $z_1(\psi)$ [i.e., to have no secular terms appear in the expression for $z_1(\psi)$] the coefficients of the fundamental harmonic entering into the right side of eq.(6.36) must be equated to zero.

By so equating, we obtain the equation

$$\left.\begin{aligned} h_n(a)=A_1(a)=0,\\ \omega_1=-\frac{g_1(a)}{2a\omega},\end{aligned}\right\} \quad (6.37)$$

determining a and ω_1. Equation (6.36) then assumes the form

$$\omega^2\left(\frac{d^2z_1}{d\psi^2}+z_1\right)=g_0(a)+\sum_{n=2}^{\infty}[g_n(a)\cos n\psi+h_n(a)\sin n\psi]. \quad (6.38)$$

By solving it we find

$$z_1(\psi)=a_1\cos\psi+\frac{g_0(a)}{\omega^2}+\frac{1}{\omega^2}\sum_{n=2}^{\infty}\frac{g_n(a)\cos n\psi+h_n(a)\sin n\psi}{1-n^2}, \quad (6.39)$$

where a_1 is an arbitrary constant which must be determined from the condition of periodicity of $z_2(\psi)$.

We will demonstrate the method for determining a_1. For this purpose, the expression for $z_1(\psi)$ is represented in the form

$$z_1(\psi)=a_1\cos\psi+u(a,\psi), \quad (6.40)$$

where

$$u(a,\psi)=\frac{g_0(a)}{\omega^2}+\frac{1}{\omega^2}\sum_{n=2}^{\infty}\frac{g_n(a)\cos n\psi+h_n(a)\sin n\psi}{1-n^2}$$

is a known periodic function of ψ.

On substituting the value of $z_1(\psi)$ from eq.(6.40) in the third equation of the system (6.34), we get

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only simple roots.

From the second equation we find ω_2:

$$\omega_2 = -\frac{1}{a}\left[\frac{v_1(a) + a_1 g_1'(a)}{2\omega} + \omega_1 a_1\right]. \tag{6.48}$$

Equation (6.45) may now be written in the form

$$\omega^2\left(\frac{d^2 z_2}{d\psi^2} + z_2\right) = (v_0(a) + a_1 g_0'(a)) + \sum_{n=2}^{\infty}\left\{\left(v_n(a) + a_1 g_n'(a)\right)\cos n\psi + \left(w_n(a) + a_1 h_n'(a)\right)\sin n\psi\right\}. \tag{6.49}$$

The solution of this equation will be

$$z_2(\psi) = a_2\cos\psi + \frac{1}{\omega^2}\left(v_0(a) + a_1 g_0'(a)\right) + \frac{1}{\omega^2}\sum_{n=2}^{\infty}\left\{\left(v_n(a) + a_1 g_n'(a)\right)\cos n\psi + \left(w_n(a) + a_1 h_n'(a)\right)\sin n\psi\right\}\frac{1}{1-n^2}, \tag{6.50}$$

where a_2 is an undetermined constant, which will be determined from the condition of periodicity of the function $z_2(\psi)$, etc.

Continuing this process, we may construct approximate solutions for the stationary state, to any predetermined degree of accuracy.

For example, in the second approximation, eqs.(6.31), (6.39) and (6.37), will yield

$$x = (a + \varepsilon a_1)\cos(\bar{\omega}t + \varphi) + \frac{\varepsilon}{\omega^2} g_0(a) + \frac{\varepsilon}{\omega^2}\sum_{n=2}^{\infty}\frac{g_n(a)\cos n(\bar{\omega}t + \varphi) + h_n(a)\sin n(\bar{\omega}t + \varphi)}{1 - n^2}, \tag{6.51}$$

where

$$\bar{\omega} = \omega + \varepsilon\frac{g_1(a)}{2\omega a}, \tag{6.52}$$

while the amplitude must be determined from the equation

$$h_n(a) = 0. \tag{6.53}$$

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By comparing the expression for x obtained in eq.(6.51) with the expression found earlier for the stationary state in eq.(6.29), we note that the only difference between these two equations lies in the fact that the amplitude of the first harmonic in eq.(6.29) is equal to a, where a is the root of the first equation of the system (6.30), while the amplitude of the first harmonic in eq.(6.51) is equal to $a + \varepsilon a_1$. This divergence, however, is in complete agreement with our remarks on this subject in Section 1.

Section 7. Equivalent Linearization of Nonlinear Oscillatory Systems

As indicated above, equations of the first approximation in most cases lead to the same qualitative results as equations of higher approximation.

In view of this, and also in view of the complexity of the calculations involved in operations with equations of higher approximations, it is usually expedient to restrict the computation to equations of the first approximation.

These equations allow a very simple physical interpretation and can be formed without first setting up the original exact differential equation, for example of the type of eq.(1.1).

In the present Section we will discuss the question of the interpretation of equations of first approximation. For this purpose, let us write the fundamental differential equation of the oscillatory system in the form

$$m\frac{d^2x}{dt^2} + kx = \varepsilon f\left(x, \frac{dx}{dt}\right), \tag{7.1}$$

where m and k are positive.

As has been established, the solution of eq.(7.1) in first approximation may be presented in the form

$$x = a\cos\psi. \tag{7.2}$$

Here, the amplitude a and the total phase ψ must satisfy the equations

$$\frac{da}{dt} = -\frac{\varepsilon}{2\pi\omega m}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi, \tag{7.3}$$

$$\frac{d\psi}{dt} = \omega_e(a),$$

where

$$\omega^2 = \frac{k}{m},$$

$$\omega_e^2(a) = \omega^2 - \frac{\varepsilon}{\pi m a}\int\limits_0^{2\pi} f(a\cos\psi, \ -a\omega\sin\psi)\cos\psi\, d\psi.$$

We recall that the first-approximation equation (7.2) is the fundamental harmonic of the approximate solution (1.4), satisfying the original equation (7.1) with an accuracy to quantities of the order of smallness of ε^m, while the amplitude a is, by hypothesis, the total amplitude of the fundamental harmonic.

Having noted this, let us introduce in the consideration of the function, the amplitudes $k_e(a)$ and $\lambda_e(a)$, defined as follows:

$$\left.\begin{aligned} \lambda_e(a) &= \frac{\varepsilon}{\pi a\omega}\int\limits_0^{2\pi} f(a\cos\psi, \ -a\omega\sin\psi)\sin\psi\, d\psi, \\ k_e(a) &= k - \frac{\varepsilon}{\pi a}\int\limits_0^{2\pi} f(a\cos\psi, \ -a\omega\sin\psi)\cos\psi\, d\psi. \end{aligned}\right\} \tag{7.4}$$

Then the equation of first approximation (7.3) may be written in the form

$$\begin{aligned} \frac{da}{dt} &= -\frac{\lambda_e(a)}{2m}a, \\ \frac{d\psi}{dt} &= \omega_e(a), \quad \omega_e^2(a) = \frac{k_e(a)}{m}. \end{aligned} \tag{7.5}$$

Let us now differentiate eq.(7.2) for the first approximation. Taking eq.(7.5) into consideration, we have:

$$\frac{dx}{dt} = -a\omega_e(a)\sin\psi - \frac{\lambda_e(a)}{2m}a\cos\psi. \tag{7.6}$$

Differentiating eq.(7.6) a second time, we can prove that

$$\frac{d^2x}{dt^2} = -a\omega_e^2(a)\cos\psi + \frac{\lambda_e(a)}{m}a\omega_e(a)\sin\psi + \frac{\lambda_e^2(a)}{4m^2}a\cos\psi + \\ + \frac{\lambda_e(a)}{2m}a^2\frac{d\omega_e(a)}{da}\sin\psi + \frac{d\lambda_e(a)}{da}\frac{a}{2m}\frac{\lambda_e(a)}{2m}a\cos\psi =$$

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$$= -\frac{k_e(a)}{m}x - \frac{\lambda_e(a)}{m}\frac{dx}{dt} - \frac{\lambda_e^2(a)}{4m^2}x + \tag{7.7}$$
$$+ \frac{\lambda_e(a)}{2m}a^2\frac{d\omega_e(a)}{da}\sin\psi + \frac{1}{2m}\frac{d\lambda_e(a)}{da}a\frac{\lambda_e(a)}{2m}x.$$

We may then, on the basis of eq.(7.4), write eq.(7.7) in the form

$$m\frac{d^2x}{dt^2} + \lambda_e(a)\frac{dx}{dt} + k_e(a)x = O(\varepsilon^2), \tag{7.8}$$

where $O(\varepsilon^2)$ is a quantity of the order of smallness of ε^2.

This shows that the first-approximation equation (7.2) under consideration will satisfy, with an accuracy to quantities of the order of smallness of ε^2, a linear differential equation of the form

$$m\frac{d^2x}{dt^2} + \lambda_e(a)\frac{dx}{dt} + k_e(a)x = 0. \tag{7.9}$$

Thus, in first approximation, the oscillations of the nonlinear oscillatory system under study are equivalent [with an accuracy to quantities of the order of smallness of ε^2, i.e., with an accuracy to quantities rejected in constructing the equations of first approximation (7.3)] to oscillations of a certain linear oscillatory system possessing a coefficient of damping $\lambda_e(a)$ and a coefficient of elasticity $k_e(a)$.

In view of this, $\lambda_e(a)$ will be denoted as the equivalent coefficient of damping and $k_e(a)$ the equivalent coefficient of elasticity, while the linear oscillatory system itself [which is described by eq.(7.9)] will be called the equivalent system.

A comparison of eq.(7.9) with eq.(7.1) shows that eq.(7.9) is obtained from eq.(7.1) by replacing the nonlinear term

$$F = \varepsilon f\left(x, \frac{dx}{dt}\right) \tag{7.10}$$

by the linear term

$$F_e = -\left[k_1(a)x + \lambda_e(a)\frac{dx}{dt}\right], \tag{7.11}$$

where $k_1(a) = k_e(a) - k$.

We note further that the expression

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$$\delta_e(a) = \frac{\lambda_e(a)}{2m}$$

represents the damping decrement of the equivalent linear system, while

$$\omega_e(a) = \sqrt{\frac{k_e(a)}{m}}$$

represents the natural frequency of oscillation of this system.

Consequently, the equations of first approximation (7.5) may be formally set up in the following manner:

Let us linearize the oscillatory system under consideration by replacing the nonlinear force (7.10) in the fundamental equation (7.1) by the linear force (7.11), in which

$$\left.\begin{aligned} \lambda_e(a) &= \frac{\varepsilon}{\pi a \omega} \int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi, \\ k_1(a) &= -\frac{\varepsilon}{\pi a} \int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\, d\psi. \end{aligned}\right\} \tag{7.12}$$

For the resultant equivalent linear system of mass m, having the damping coefficient $\lambda_e(a)$ and the coefficient of elasticity $k_e(a) = k + k_1(a)$, we find, by the usual method, the damping decrement $\delta_e(a)$ and the frequency of the natural oscillations $\omega_e(a)$, rejecting in this case all quantities of the second order of smallness. Then we obtain

$$\delta_e(a) = \frac{\lambda_e(a)}{2m}, \quad \omega_e^2(a) = \frac{k_e(a)}{m}. \tag{7.13}$$

In forming the expression for the damping decrement and the frequency, we use the generally known formulas for linear oscillatory systems

$$\frac{da}{dt} = -\delta a, \qquad \frac{d\psi}{dt} = \omega, \tag{7.14}$$

which disclose the fact that the damping decrement is the logarithmic derivative of the amplitude taken with reversed sign, and that the frequency ω is the angular velocity of rotation of the total phase of the oscillation.

If in eq.(7.14) we substitute the values of δ and ω from eqs.(7.13) and (7.4),

we obtain equations agreeing with the equations of first approximation (7.3) previously derived.

This formal method of setting up the equations of first approximation will be called the method of linearization.

In using this method, the question arises as to the reason why, from the physical point of view, in linearization - or more exactly - in the substitution of the nonlinear force (7.10) by the equivalent linear force (7.11) - the coefficients $k_1(a)$ and $\lambda_e(a)$ should assume precisely the values given by eq.(7.12) instead of any other values. In view of this, it is necessary to solve the question of the physical interpretation for these formulas.

We point out, that the values of the equivalent damping coefficient, corresponding to eq.(7.12), were obtained under the assumption that the mean power (over the period of oscillation) developed by the actual force of eq.(7.10) is equal to that developed by the equivalent force of eq.(7.11). In this case, by equating the expressions for both powers, we must neglect quantities of the order of smallness of ε^2, since the equations of first approximation are accurate only to quantities of precisely this order of smallness.

Since the work performed by the force $k_e(a)x$ is proportional to a displacement, which, during the period of oscillation, is equal to zero, equating the powers developed by the forces (7.10) and (7.11) will yield

$$\varepsilon \int_0^T f\left(x, \frac{dx}{dt}\right) \frac{dx}{dt}\, dt = -\lambda_e(a) \int_0^T \left(\frac{dx}{dt}\right)^2 dt, \tag{7.15}$$

where T is the period of oscillation. It follows likewise from eq.(7.15) that $\lambda_e(a)$ must be a quantity of the first order of smallness.

At an accuracy to quantities of the order of smallness of ε we may assume, over a time interval of the order of $\frac{2\pi}{\omega}$, that

$$x = a\cos(\omega t + \theta), \quad \frac{dx}{dt} = -a\omega \sin(\omega t + \theta), \tag{7.16}$$

where a and θ are constant throughout the course of this interval. In the same approximation ω will be the frequency of the oscillations and $T = \frac{2\pi}{\omega}$, the period.

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Substituting eq.(7.16) on both sides of eq.(7.15) and taking into consideration that $T = \frac{2\pi}{\omega}$, the following expression is obtained, with an accuracy to quantities of the second order of smallness [since both sides of eq.(7.15) contain factors that are first-order quantities, namely ε and $\lambda_e(a)$]:

$$-\varepsilon\omega\int_0^{\frac{2\pi}{\omega}} f[a\cos(\omega t+\vartheta), \; -a\omega\sin(\omega t+\vartheta)]\, a\sin(\omega t+\vartheta)\,dt = \tag{7.17}$$

$$= -\lambda_e(a)\int_0^{\frac{2\pi}{\omega}} a^2\omega^2\sin^2(\omega t+\vartheta)\,dt = -\lambda_e(a)a^2\pi\omega$$

or

$$\lambda_e(a)\,\pi\omega a^2 = \varepsilon a\int_0^{2\pi} f(a\cos\psi, \; -a\omega\sin\psi)\sin\psi\,d\psi.$$

This exactly yields the value of the coefficient of $\lambda_e(a)$, which is determined from eq.(7.12).

To obtain an analogous interpretation for the other coefficient, namely for $k_1(a)$, let us make use of the concept of reactive power that is current in electrical engineering.

We will briefly review the principle of reactive power:

Let the alternating current $i(t)$ flow along a certain conductor AB, and let $E(t)$ be the difference in voltage between the terminals A and B of this conductor.

Then the active power P_a given off or absorbed in the conductor under consideration (depending on the sign) is called the quantity of work performed during the period of oscillation T, divided by the quantity T, i.e.,

$$P_a = \frac{1}{T}\int_0^T E(t)\,i(t)\,dt. \tag{7.18}$$

It is clear that the concept of active power completely corresponds to the ordinary concept of mean mechanical power; being connected with the concept of work or energy, it has an entirely real physical significance.

In electrical engineering, however, it is customary to introduce into the consideration, besides the concept of active power, which does possess direct physical

significance, the somewhat artificial concept of reactive power.

Reactive power is the term applied to the quantity

$$P_r = \frac{1}{T}\int_0^T E(t)\, i^*(t)\, dt, \tag{7.19}$$

where the function $i^*(t)$ represents a current of the same form as the current $i(t)$, but with a 90° phase lag with respect to the current $i(t)$ or, differently expressed,

$$i^*(t) = i\left(t - \frac{T}{4}\right). \tag{7.20}$$

In view of the obvious and direct analogy between mechanical and electrotechnical oscillations, it will be expedient to use the concept of reactive power also for mechanical vibrations.

Let us assume, for example, that a certain body, under the action of a certain force $F(t)$, executes periodic vibrations.

Let T be the period of the vibrations and $x(t)$ the displacement of the body. Since the active power in this case corresponds to the mean mechanical power

$$\frac{1}{T}\int_0^T F(t)\, x'(t)\, dt, \tag{7.21}$$

then, reasoning by analogy with the preceding, it is natural to call an expression of the form

$$\frac{1}{T}\int_0^T F(t)\, x'\left(t - \frac{T}{4}\right) dt. \tag{7.22}$$

reactive power.

Adopting this definition and returning to the question of interpreting the values of eq.(7.12) for the equivalent coefficient $k_1(a)$, we will show that precisely this value is obtained when first postulating the equality of the reactive powers (again with an accuracy to quantities of the order of smallness of ε^2) developed by the real and equivalent forces. Indeed, by equating the expressions for the two reactive powers, we obtain:

$$\varepsilon \frac{1}{T}\int_0^T f[x(t),\ x'(t)]\,x'\left(t-\frac{T}{4}\right)dt =$$
$$= -\frac{1}{T}\int_0^T [k_1(a)\,x(t)+\lambda_e(a)\,x'(t)]\,x'\left(t-\frac{T}{4}\right)dt. \tag{7.23}$$

Since $\lambda_e(a)$ is a quantity of the first order of smallness, it is obvious that $k_1(a)$ will be of the same order of smallness with respect to ε.

For this reason, by substituting the following formulas in the expression for the reactive power:

$$x = a\cos(\omega t+\vartheta),\quad \frac{dx}{dt} = -a\omega\sin(\omega t+\vartheta),\quad T=\frac{2\pi}{\omega},$$

which are exact to quantities of the first order of smallness, we have, with the required accuracy

$$\varepsilon\frac{1}{T}\int_0^T f[x(t),\ x'(t)]\,x'\left(t-\frac{T}{4}\right)dt =$$
$$= \frac{a\omega\varepsilon}{2\pi}\int_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\cos\psi\,d\psi,$$

$$\frac{1}{T}\int_0^T [k_1(a)\,x(t)+\lambda_e(a)\,x'(a)]\,x'\left(t-\frac{T}{4}\right)dt = \frac{a^2\omega k_1(a)}{2},$$

whence, on the basis of eq.(7.23), we obtain

$$k_1(a) = -\frac{\varepsilon}{\pi a}\int_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\cos\psi\,d\psi,$$

i.e., the same expression as from eq.(7.12).

Thus, to summarize, we see that, on using the method of linearization, the parameters (equivalent coefficients) of the equivalent linear force

$$F_e = -\left[k_1(a)\,x+\lambda_e(a)\frac{dx}{dt}\right], \tag{7.24}$$

replacing the nonlinear force

$$F = \varepsilon f\left(x,\ \frac{dx}{dt}\right), \tag{7.25}$$

may be determined by equating the expressions for the active and reactive power de-

veloped by the forces (7.24) and (7.25) under harmonic oscillations:

$$x = a\cos(\omega t + \vartheta),$$

where ω is the frequency in the "zero" approximation.

This method of determining the equivalent coefficients will be called the principle of power or the principle of energetic balance.

We shall now present another and simpler method of determining the equivalent coefficients.

For this purpose, let us substitute the values of x and of $\frac{dx}{dt}$ given by eq.(7.16) in eqs.(7.24) and (7.25).

For the harmonic oscillation of eq.(7.16), the linear equivalent force F_e will likewise be a harmonic function of time, with the frequency ω. Denoting the amplitude and the phase of F_e by J_e and φ_e, respectively, we have

$$F_e = J_e \cos(\omega t + \varphi_e). \tag{7.26}$$

The nonlinear force will, generally speaking, be a periodic function of time, consisting of various harmonics with frequencies of the form $n\omega$, where n = 1, 2,...

Assume that

$$J\cos(\omega t + \varphi) \tag{7.27}$$

is its fundamental harmonic. Then, on equating the amplitude and the phase

$$-k_1(a)\,a\cos(\omega t + \vartheta) + \omega\lambda_e(a)\,a\sin(\omega t + \vartheta),$$

of the equivalent force (7.26) and of the fundamental harmonic of the nonlinear force (7.27), we obtain two equations which give, for the parameters $k_1(a)$ and $\lambda_e(a)$, precisely the values that are given by eq.(7.12).

Indeed, in the expanded form, under harmonic oscillation, the equivalent linear force will be

$$J_e = J, \quad \varphi_e = \varphi \tag{7.28}$$

and the fundamental harmonic of the nonlinear force will become

$$\left\{\frac{1}{\pi}\int_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\cos\psi\, d\psi\right\}\cos(\omega t + \vartheta) +$$

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$$+\left\{\frac{1}{\pi}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi\right\}\sin(\omega t+\theta). \qquad (7.29)$$

On equating the two harmonics of eqs.(7.28) and (7.29), we obtain (in practice it is simpler to equate not the amplitudes and phases, but the coefficients of the sines and cosines in the expressions for the first harmonics, respectively):

$$\left.\begin{aligned} ak_1(a) &= -\frac{1}{\pi}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\, d\psi, \\ \omega\lambda_e(a)a &= \frac{1}{\pi}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi, \end{aligned}\right\} \qquad (7.30)$$

whence, for $k_1(a)$ and $\lambda_e(a)$ we obtain the very same values that are given by equation (7.12).

We will call this latter method the principle of harmonic balance.

It is not difficult to establish that the principles of energetic and harmonic balance are essentially equivalent.

For this it is necessary to note that the expressions for the power

$$\frac{1}{T}\int_0^T F(t)\,x'(t)\,dt, \quad \frac{1}{T}\int_0^T F(t)\,x'\left(t-\frac{T}{4}\right)dt,$$

developed by a certain periodic force F(t) (with the period $T = \frac{2\pi}{\omega}$) for the harmonic oscillations

$$x = a\cos(\omega t+\theta),$$

will depend only on the fundamental harmonic F(t).

Thus, if the fundamental harmonics (the harmonics with the frequency ω) of the given forces are equal, then the power developed by them under harmonic oscillations (of the frequency ω) are also equal, and vice versa.

This fact is responsible for the substantial equivalence of the two above methods for determining the parameters of the equivalent linear system in the methods of linearization (the principle of harmonic balance and the principle of energetic balance).

We note now that there is no need to set up the differential equation of the

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oscillations first, and only then to linearize the nonlinear expressions occurring in it.

In many cases, especially for more or less complex oscillatory systems, it may be more convenient, before setting up the differential equation, to consider the diagram of the oscillatory system directly and to replace the nonlinear elements by the equivalent linear elements (for example, using the principle of harmonic balance), and only then to find the expressions for the frequency $\omega_e(a)$ and the decrement $\delta_e(a)$, starting from the generally known and classical formulas of the theory of linear oscillations.

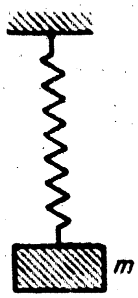

Fig. 37

The basic condition for the admissibility of this type of method of equivalent linearization evidently is that the oscillations be approximately harmonic.

As an example, let us consider a body of mass m, suspended from a spring and executing approximately harmonic oscillations (Fig. 37). Let the relation between the elastic force of the spring F and its elongation x be nonlinear, and expressed by the following equation

$$F = f(x). \tag{7.31}$$

Then, for the harmonic oscillations

$$x = a\cos(\omega t + \theta)$$

the fundamental harmonic of the force of elasticity will be

$$\frac{\cos(\omega t + \theta)}{\pi}\int_0^{2\pi} f(a\cos\varphi)\cos\varphi\, d\varphi.$$

Making use of the above-discussed principle of harmonic balance, we may replace the actual nonlinear spring by an equivalent linear spring with the coefficient of elasticity

$$k_e(a) = \frac{1}{\pi a}\int_0^{2\pi} f(a\cos\varphi)\cos\varphi\, d\varphi, \tag{7.32}$$

after which, according to well-known formulas, we find the frequency of the linearized system

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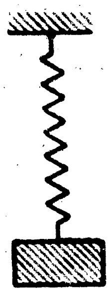

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after which, according to well-known formulas, we find the frequency of the linearized system

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$$\omega_e(a)=\sqrt{\frac{k_e(a)}{m}}. \tag{7.33}$$

If the oscillatory system under consideration, consisting of a body of mass m and a nonlinear spring, is suspended from a linear spring with the coefficient of elasticity c (Fig. 38), then, assuming the oscillations to be approximately harmonic (c being sufficiently great) we obtain, according to the principle of harmonic balance, the following formula for the oscillation frequency:

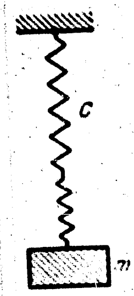

Fig. 38

$$\omega_e(a)=\sqrt{\frac{c+k(a)}{m}}, \tag{7.34}$$

or, with an accuracy to quantities of the first order of smallness,

$$\omega_e(a)=\sqrt{\frac{c}{m}}\left(1+\frac{1}{2}\frac{k(a)}{c}\right). \tag{7.35}$$

Let us now assume that our oscillatory system undergoes, in its oscillation, a certain weak damping influence of a nonlinear type, depending only on the velocity

$$\Phi=\Phi\left(\frac{dx}{dt}\right). \tag{7.36}$$

Then, assuming that the form of these oscillations remains close to harmonic, we obtain the following expression for the fundamental harmonic of the damping force

$$\frac{\sin(\omega t+\theta)}{\pi}\int_0^{2\pi}\Phi(-a\omega\sin\varphi)\sin\varphi\,d\varphi.$$

On the basis of the principle of harmonic balance, this actual force may be replaced by the equivalent linear damping force

$$\Phi_e=\lambda_e(a)\frac{dx}{dt},$$

where the coefficient of friction is expressed by the following formula:

$$\lambda_e(a)=-\frac{1}{a\omega\pi}\int_0^{2\pi}\Phi(-a\omega\sin\varphi)\sin\varphi\,d\varphi, \tag{7.37}$$

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after which, according to conventional formulas, we find the following expression for the damping decrement:

$$\delta_e(a) = \frac{\lambda_e(a)}{2m}. \tag{7.38}$$

We note that the presence of friction, at the degree of accuracy adopted, will have no effect on the frequency of the oscillations, since $\left(\frac{\lambda_e(a)}{cm}\right)^2$ is a quantity of the second order of smallness.

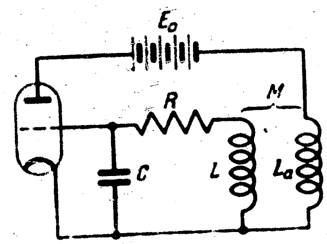

Fig. 39

Let us consider, as a second example, the electronic oscillator designed according to the circuit diagram in Fig. 39. Let the resistance R, connected in series, be sufficiently small and, consequently, let the oscillatory circuit, located in the grid circuit of this diagram be slightly damped.

Let i be the current in the oscillatory circuit. Then we have

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = M\frac{di_a}{dt}, \tag{7.39}$$

where i_a is the plate current, depending on the control voltage E (in the absence of grid current),

$$i_a = f(E). \tag{7.40}$$

The control voltage in the electronic oscillator is composed of the constant voltage E_0 and the alternating control voltage e, induced by the oscillatory circuit.

Thus, the relation between the plate current and the alternating component of the alternating component of the control voltage will be

$$i_a = f(E_0 + e). \tag{7.41}$$

If e varies by the law

$$e = a\cos(\omega t + \theta),$$

then the fundamental harmonic of the plate current will be

$$\frac{\cos(\omega t+\theta)}{\pi}\int_0^{2\pi} f(E_0+a\cos\varphi)\cos\varphi\,d\varphi;$$

Therefore, on the basis of the principle of harmonic balance, we may replace the nonlinear relation (7.41) by the equivalent linear relation

$$i_a = Se, \tag{7.42}$$

where the parameter S, the "mean grid-plate transconductance of the tube" has the form

$$S(a)=\frac{1}{\pi a}\int_0^{2\pi} f(E_0+a\cos\varphi)\cos\varphi\,d\varphi. \tag{7.43}$$

Equation (7.39) may now be presented in the form

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int i\,dt = MS\frac{de}{dt}. \tag{7.44}$$

On the other hand, starting from the diagram of Fig. 39, we may write

$$e=\frac{1}{C}\int i\,dt + D\left(M\frac{di}{dt}-L_a\frac{di_a}{dt}\right).$$

In the case where the grid through of the tube is very small, i.e., where D is close to zero, we may take

$$e=\frac{1}{C}\int i\,dt. \tag{7.45}$$

On substituting eq. (7.45) in eq. (7.44), we obtain

$$LC\frac{d^2e}{dt^2}+(RC-MS)\frac{de}{dt}+e=0,$$

whence, according to conventional formulas, we find the expressions for the natural frequency and the damping decrement

$$\left.\begin{aligned}\omega &= \frac{1}{\sqrt{LC}},\\ \delta_e(a) &= \frac{RC-MS(a)}{2LC}.\end{aligned}\right\} \tag{7.46}$$

These simple examples show the manner of applying the method of linearization directly to the circuit diagram of the given oscillatory system. In this case, it

is obvious that the nonlinear elements of the system are linearized independently of the other parameters of the system, whose role in linearization finally consists in ensuring an approximately harmonic nature of the oscillations.

Let us emphasize again the fact that the equivalent linear elements differ substantially from true linear elements in that their parameters - the equivalent coefficients - are not constant, but are certain definite functions of the amplitude of oscillation.

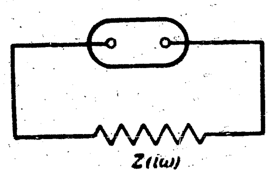

Fig. 40

The advantage of the method of equivalent linearization in the effective construction of the equations of first approximation becomes particularly plain in the case of complex oscillatory systems for which even setting up the fundamental differential equations of the oscillatory process, to say nothing of operations on them, may cause difficulties.

As an application of the methods of equivalent linearization to systems with many degrees of freedom* let us consider the oscillatory circuit schematically shown in Fig. 40, consisting of a linear part with the complex resistance $Z(i\omega)$ and a nonlinear element with the voltage-current characteristic

$$V = f(I).$$

Then, the corresponding differential equation will be

$$Z(p)I = f(I), \quad p = \frac{d}{dt}. \tag{7.47}$$

Let us assume that the parameters of the system are such that the periodic quasi-harmonic oscillations

$$I \approx a\cos(\omega t + \psi). \tag{7.48}$$

are excited in it. Then we may use the methods of equivalent linearization and replace the nonlinear elements in the first approximation by a linear element with the characteristics

$$V = SI,$$

* Such methods may be applied to the study of systems of a more general type. Cf, for example, Ye.P.Popov, (Bibl.48).

where

$$S(a) = \frac{1}{\pi a}\int_0^{2\pi} f(a\cos\psi)\cos\psi\, d\psi.$$

For stationary oscillations, we then find an equation of the form

$$Z(i\omega) = S(a),$$

whence, by separating the real and imaginary parts of the complex resistance

$$Z(i\omega) = X(\omega) + iY(\omega),$$

we obtain

$$\left.\begin{aligned} X(\omega) &= S(a), \\ Y(\omega) &= 0. \end{aligned}\right\} \tag{7.49}$$

One of the resultant equations serves to determine the frequency, the other to determine the amplitude of oscillations.

The method of equivalent linearization may be refined in such a way as to obtain not only the equation of the first approximation but also those of higher approximations.

Let us construct the equations of second approximation. For this purpose, let us refine eq.(7.48) by including the higher harmonics and a constant term.

In first approximation, the voltage on the nonlinear element will be

$$V = f(a\cos\theta), \quad \theta = \omega t + \psi. \tag{7.50}$$

By expanding this expression into a Fourier series, we obtain

$$V = \sum_{(n \geqslant 0)} f_n(a)\cos n\theta,$$

where, as usual,

$$f_0(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a\cos\theta)\, d\theta,$$

$$f_n(a) = \frac{1}{\pi}\int_0^{2\pi} f(a\cos\theta)\cos n\theta\, d\theta.$$

We now note that the harmonic component of the voltage

$$f_n(a)\cos(n\omega t + n\psi), \quad (n = 0,\ 2,\ 3,\ \ldots)$$

produces, in the linear element, the current

$$Z^{-1}(p) f_n(a)\cos(n\omega t + n\psi).$$

Let us introduce the absolute value and the phase of the complex resistance

$$Z(i\Omega) = R(\Omega)\,e^{i\varphi(\Omega)}.$$

Then,

$$Z^{-1}(p) f_n(a) \cos(n\omega t + n\psi) = \frac{f_n(a)}{R(n\omega)} \cos(n\omega t + n\psi - \varphi(n\omega)).$$

In this way, the more exact variant of eq.(7.48) will become

$$I \approx a \cos\vartheta + \sum_{(n=0,\,2,\,3,\,\ldots)} \frac{f_n(a)}{R(n\omega)} \cos(n\vartheta - \varphi(n\omega)). \qquad (7.51)$$

Whence, among other things, a criterion of the applicability of this method is immediately obtained. It is necessary that the term a cos θ be the principal term, and the other terms only correction terms. For this reason and to make this method applicable, it is necessary that

$$\left|\frac{f_n(a)}{R(n\omega)}\right| \ll a, \quad (n = 0,\, 2,\, 3,\, \ldots).$$

Let us make use of eq.(7.51) to render eq.(7.50) more precise.

We have

$$V = f(a\cos\vartheta + \xi),$$

where

$$\xi = \sum_{(n=0,\,2,\,3,\,\ldots)} \frac{f_n(a)}{R(n\omega)} \cos(n\vartheta - \varphi(n\omega)),$$

However, since ξ must be small by comparison with the first term, the voltage V will be

$$V = f(a\cos\vartheta) + \xi f'(a\cos\vartheta).$$

By expanding the resultant expression into a Fourier series, we find

$$V = \sum_{(n>0)} f_n(a)\cos n\vartheta + \\ + \Big\{\sum_{(n>1)} [\Phi_n(a)\cos n\vartheta + G_n(a)\sin n\vartheta] + \Phi_0(a)\Big\}, \qquad (7.52)$$

where the sum within the braces yields the expansion into a Fourier series, of the function

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$$\xi f'(a\cos\theta).$$

We will be particularly interested in the values of $\Phi_1(a)$ and $G_1(a)$. We therefore present the corresponding formules

$$\Phi_1(a)=\frac{1}{\pi}\int_0^{2\pi}\xi f'(a\cos\theta)\cos\theta\,d\theta,$$

$$G_1(a)=\frac{1}{\pi}\int_0^{2\pi}\xi f'(a\cos\theta)\sin\theta\,d\theta.$$

On considering the stationary oscillations again, we equate the first harmonic of the voltage from eq.(7.52) to the first harmonic of the voltage across the linear element

$$Z(i\omega)\,a\cos(\omega t+\psi)=R(\omega)\,a\cos(\omega t+\psi+\varphi(\omega)),$$

and then find

$$f_1(a)\cos(\omega t+\psi)+\Phi_1(a)\cos(\omega t+\psi)+G_1(a)\sin(\omega t+\psi)=$$
$$=R(\omega)\,a\cos(\omega t+\psi+\varphi(\omega)),$$

whence we obtain the refined equations of harmonic balance

$$R(\omega)\,a\cos\varphi(\omega)=f_1(a)+\Phi_1(a),$$
$$R(\omega)\,a\sin\varphi(\omega)=-G_1(a),$$

or

$$\left.\begin{aligned} X(\omega)&=\frac{f_1(a)+\Phi_1(a)}{a},\\ Y(\omega)&=-\frac{G_1(a)}{a}.\end{aligned}\right\}\qquad(7.53)$$

Let us now simplify the expressions for $\Phi_1(a)$ and $G_1(a)$. We have

$$\Phi_1(a)=\sum_{(n\neq 1)}\frac{f_n(a)}{R(n\omega)}\frac{1}{\pi}\int_0^{2\pi}f'(a\cos\theta)\cos\theta\cos(n\theta-\varphi(n\omega))\,d\theta=$$

$$=\frac{1}{2}\sum_{(n\neq 1)}\frac{\frac{df_n^2(a)}{da}}{R(n\omega)}\cos\varphi(n\omega),$$

$$O_1(a) = \sum_{(n \neq 1)} \frac{f_n(a)}{R(n\omega)} \frac{1}{\pi} \int_0^{2\pi} f'(a \cos \theta) \sin \theta \sin (n\theta - \varphi(n\omega))\, d\theta =$$

$$= \sum_{(n \neq 1)} \frac{f_n(a)}{R(n\omega)} \cos \varphi(n\omega) \frac{1}{\pi} \int_0^{2\pi} f'(a \cos \theta) \sin \theta \sin n\theta \, d\theta .$$

However,

$$\frac{1}{\pi} \int_0^{2\pi} f'(a \cos \theta) \sin \theta \sin n\theta \, d\theta = -\frac{1}{a\pi} \int_0^{2\pi} \frac{\partial f(a \cos \theta)}{\partial \theta} \sin n\theta \, d\theta =$$

$$= \frac{n}{a\pi} \int_0^{2\pi} f(a \cos \theta) \cos n\theta \, d\theta = \frac{n f_n(a)}{a},$$

i.e.,

$$O_1(a) = \sum_{(n > 2)} \frac{n f_n^2(a)}{aR(n\omega)} \cos \varphi(n\omega).$$

Thus the refined equations of harmonic balance for stationary oscillations will be

$$\left.\begin{aligned} X(\omega) &= \frac{f_1(a)}{a} + \frac{1}{2} \sum_{(n=0,2,3,\ldots)} \frac{\dfrac{df_n^2(a)}{da}}{aR(n\omega)} \cos \varphi(n\omega), \\ Y(\omega) &= -\sum_{(n > 2)} \frac{n f_n^2(a)}{a^2 R(n\omega)} \cos \varphi(n\omega). \end{aligned}\right\} \qquad (7.54)$$

On comparing them with the equations of the first approximation of eq. (7.49), we see that the influence of first harmonics of the oscillations is reflected here. The equation (7.54) so obtained can also be used for a more detailed elucidation of the limits of applicability of the equations of first approximation.

We note also that these results could have been obtained by the method of asymptotic expansions. For this purpose, it is expedient to represent the fundamental equation of the oscillatory process in eq. (7.47), for example, in the form

$$\left[(p^2 + \omega_0^2) Q(p) + \varepsilon S(p)\right] I = \varepsilon f(I). \qquad (7.55)$$

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CHAPTER II

THE METHOD OF THE PHASE PLANE

Section 8. Paths on the Phase Plane. Singular Points

The above-described asymptotic methods are limited in their application by the requirement that a small parameter be present in the equation. In many cases, however, we have to do with equations of a more general type, to which these methods are inapplicable.

If the equations describing the motion of the dynamic system under study can be reduced to the form

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \tag{8.1}$$

then qualitative methods of study can be applied to them. All equations considered in this book can be reduced to equations of the type of eq.(8.1). In addition, as we shall see later, in studying oscillatory systems that are weakly nonlinear but are under the influence of external periodic forces, equations of the type of eq.(8.1) are also obtained as equations of first approximation.

For a qualitative investigation of the solutions of eq.(8.1) it is expedient to consider x, y as the coordinates of a point on a plane. This plane, as generally known, is termed the phase plane, and the point x, y the phase point. The motion $x = x(t)$, $y = y(t)$ is performed along a certain line which is called a phase path. The construction of the phase path of a given system means the construction of a curve expressing the velocity as a function of the displacement, for the assigned

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motion.

The phase plane, with the phase paths on it, immediately shows the totality of all motions that can occur in the dynamic system under consideration.

To construct the phase paths we must find the solution of the system of eq.(8.1) $x = x(t)$, $y = y(t)$, representing the equation of the phase path in a parametric form, or find the characteristics, i.e., the integral curves of the equation

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}, \qquad (8.2)$$

which directly yield the relationship between x and y.

Let us discuss the simplest cases first.

Consider the equation of a linear oscillator:

$$\frac{d^2x}{dt^2} + 2h\frac{dx}{dt} + kx = 0. \qquad (8.3)$$

Putting

$$\frac{dx}{dt} = y,$$

we bring eq.(8.3) into the form

$$\left.\begin{aligned} \frac{dy}{dt} &= -2hy - kx, \\ \frac{dx}{dt} &= y. \end{aligned}\right\} \qquad (8.4)$$

Assume that the friction is small, i.e. $h^2 < k$, $k > 0$; then the solution of the system (8.4) is written in the form

$$\left.\begin{aligned} x &= ae^{-ht}\cos(\omega_1 t + \alpha), \\ y &= -a\omega e^{-ht}\sin(\omega_1 t + \alpha + \theta), \end{aligned}\right\} \qquad (8.5)$$

where $\omega_1^2 = k - h^2$, $\theta = \tan^{-1}\frac{h}{\omega_1}$, a and α are arbitrary constants determined by the initial values.

Equation (8.5) is the equation of the phase path in parametric form. By its aid the character of the motion of the phase point on the phase plane can be

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analyzed without difficulty.

We note that the system (8.4) determines, at each point of the phase plane, a single tangent to the integral curve, except at the point $x = 0$, $y = 0$. The slope of the tangent is defined by the expression

$$\frac{dy}{dx} = \frac{-2hy - kx}{y}. \qquad (8.6)$$

At the point $x = 0$, $y = 0$, the direction of the tangent becomes indeterminate. Such points are called critical or singular points. For the simplest singular points (singular points of the first order, or elementary points) either no integral curve at all or more than one integral curve pass through the singular point.

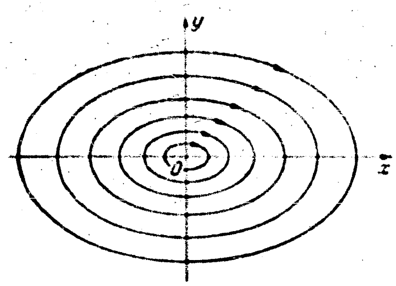

Fig. 41

Let us assume first that $h = 0$. Then the solution of eq.(8.5) assumes the form

$$\left.\begin{aligned} x &= a\cos(\omega t + \alpha), \\ y &= -a\omega\sin(\omega t + \alpha). \end{aligned}\right\} \qquad (8.7)$$

On the phase plane we obtain a family of similar ellipses (Fig. 41), and in this case, only one ellipse, corresponding to definite initial conditions, passes through each point of the phase plane.

By eliminating the time t from eq.(8.7) we obtain the equation of the family of ellipses in the form

$$\frac{x^2}{a^2} + \frac{y^2}{a^2\omega^2} = 1, \qquad (8.8)$$

which we could also have obtained by directly integrating eq.(8.6) at $h = 0$.

Not a single integral curve passes through the origin of coordinates. Such a singular point, in whose neighborhood the integral curves are closed and surround the singular point, is called a center.

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Since, in this case, all the phase paths are closed curves or ellipses, (except the path that degenerates to the point $x = 0$, $y = 0$), the motion will be periodic.

The singular point $x = 0$, $y = 0$ corresponds to the state of equilibrium in the oscillatory system under consideration.

It is entirely clear that, in the general case of eq.(8.1), the states of equilibrium of the system correspond, on the phase plane, to the points for which, simultaneously, $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$, i.e., the singular points of eq.(8.2); to the periodic motions taking place in the system there correspond, on the phase plane, the closed phase paths of eq.(8.2).

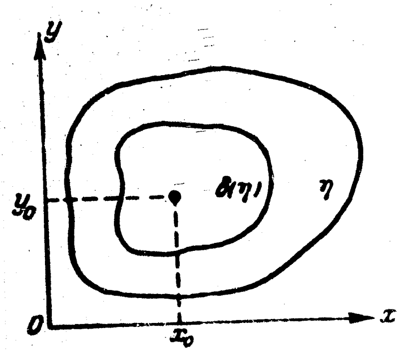

Fig.42

In the following, we will have occasion to make use of the concept of stability of the equilibrium state.

We here present only its definition, without going into detail. A state of equilibrium is stable, if, for any assigned region of allowable deviations from the equilibrium state (the region η), we are able to indicate a region $\delta(\eta)$, surrounding the state of equilibrium and possessing the property that no motion commencing within $\delta(\eta)$, ever leaves the region η (Fig.42).

Analytically, this definition of stability may be expressed as follows: The state of equilibrium $x = x_0$, $y = y_0$ is called stable, if, for any η assigned in advance, no matter how small, a $\delta(\eta)$ can be found, so that, for $t = t_0$,

$$|x(t_0) - x_0| < \delta(\eta), \quad |y(t_0) - y_0| < \delta(\eta),$$

but also for any values of t such that $t_0 < t < \infty$,

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$$|x(t)-x_0|<\eta, \quad |y(t)-y_0|<\eta.$$

It is obvious that a state of equilibrium of the type of a center is a stable state of equilibrium.

Now let $h > 0$, which corresponds to a damped oscillatory process. In this case, we obtain from eq.(8.5), on the phase plane, a family of spirals for which the origin of coordinates is the asymptotic point (Fig.43); in this case, the smaller the ratio h/ω_1, i.e., the smaller the friction, the closer will the spiral approach the form of an ellipse during the course of a single revolution. The phase velocity in this case will not vanish anywhere, except at the origin of coordinates, but will continuously decrease as the representative point approaches the origin of coordinates.

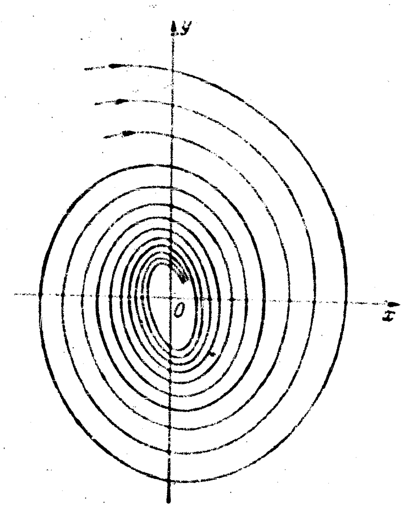

Fig.43

The phase paths correspond in this case to oscillatory but damped motions, while the singular point $x = 0$, $y = 0$ corresponds to the equilibrium position.

The singular point under consideration in this case, being the asymptotic point of all integral curves having the form of spirals, is called the focus; for $h > 0$, this focus will be stable.

Now let $h < 0$. In this case, we again obtain a family of spirals (Fig.44) but the phase point, in time, will leave the origin of coordinates. The velocity of

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motion of the representative point along the integral curve, which becomes zero at $x = 0$, $y = 0$, will monotonously increase as the point moves away from the origin of coordinates. In this case the position of equilibrium is unstable, and the singular point $x = 0$, $y = 0$ is an unstable focus.

Let us now consider the case when $h^2 > k$, which corresponds, for $h > 0$, to a

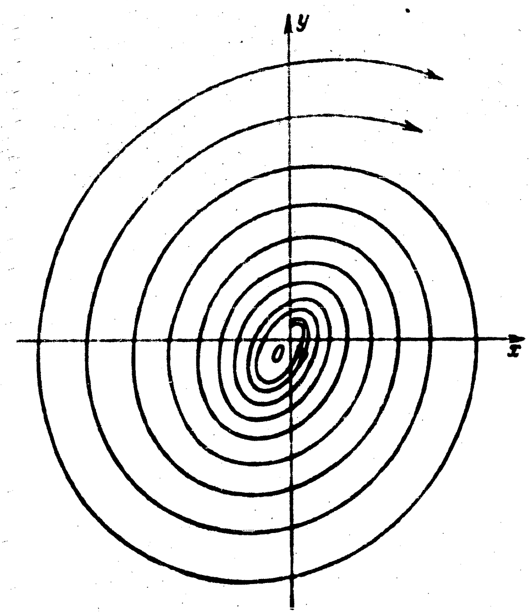

Fig. 44

damped aperiodic process. In this case, the solution of the system (8.4) may be represented in the form

$$x = C_1 e^{-q_1 t} + C_2 e^{-q_2 t}, \tag{8.9}$$

$$y = -C_1 q_1 e^{-q_1 t} - C_2 q_2 e^{-q_2 t}, \tag{8.10}$$

where we denote

$$-q_1 = -h + \sqrt{h^2 - k},$$

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$$q_2 = -h - \sqrt{h^2 - k}. \Big\} \tag{8.11}$$

To obtain the image on the phase plane, multiply eq. (8.9) first by q_1, then by q_2, and add to eq. (8.10). On raising the results so obtained to the powers q_1 and q_2, respectively, we find

$$(y + q_1 x)^{q_1} = C(y + q_2 x)^{q_2} \tag{8.12}$$

or

$$y + q_1 x = C(y + q_2 x)^{\frac{q_2}{q_1}}. \tag{8.13}$$

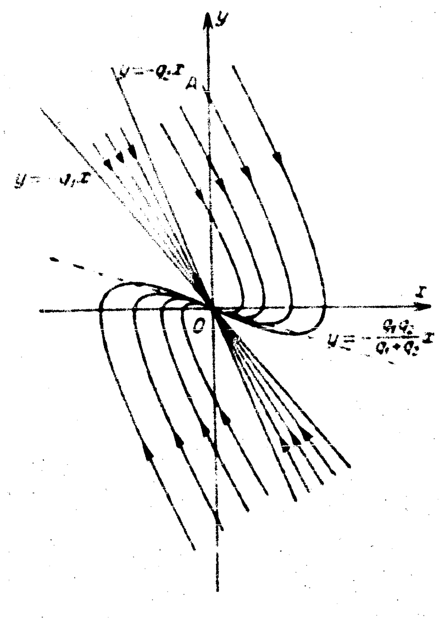

Fig. 45

On the phase plane we obtain a family of deformed parabolas (Fig. 45), which are tangent to the straight line $y = -q_1 x$ at the origin of coordinates. It is

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not difficult to establish the direction of the representative point along the integral curves. The representative point will move along the integral curves in the direction indicated on Fig.45 by the arrows, i.e. it will always approach the origin of coordinates.

The point $x = 0$, $y = 0$ will be a singular point, and all integral curves will pass through it. A singular point of this type is called a node. In the case under

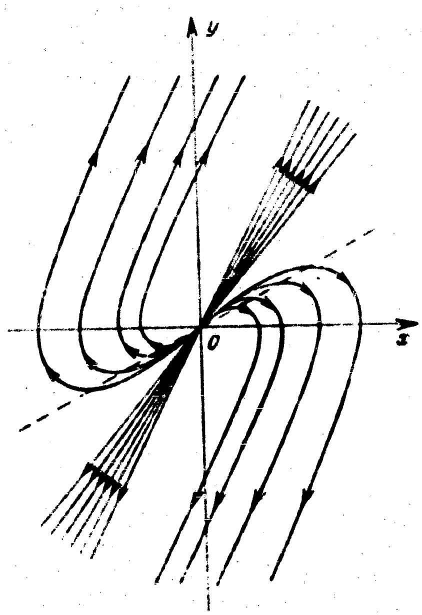

Fig. 46

consideration the equilibrium position will be stable, and a stable node will correspond to this position.

Consider now the character of the integral curves on the phase plane for the case of a high negative friction $h < 0$, $h^2 > k$. In this case, making use of eq.(8.13), a family of parabolic-type curves (Fig.46) with a singular point of the node type is obtained on the phase plane. An analysis of the motion of the repre-

sentative point, however, readily shows that this point, in its motion along any of the integral curves, tends to leave the state of equilibrium (cf. Fig. 46) in which the direction of motion is indicated by arrows). Thus, the singular point under consideration constitutes an unstable node, and, as above, the instability here is again due to the fact that $h < 0$. We note that the instability is evident from a consideration of eq. (8.9) and (8.10).

For completeness of exposition, we now present another type of phase path, which will be encountered later in the text. For this purpose, let us consider the differential equation

$$\frac{d^2x}{dt^2} - kx = 0, \tag{8.14}$$

where k is positive. We obtain an equation of the type of eq. (8.14) by considering, for example, small deflections of a pendulum alongside its upper position of unstable equilibrium.

The solution of eq. (8.14) will be

$$\left.\begin{aligned} x &= C_1 e^{\sqrt{k}t} + C_2 e^{-\sqrt{k}t}, \\ y &= C_1 \sqrt{k} e^{\sqrt{k}t} - C_2 \sqrt{k} e^{-\sqrt{k}t}. \end{aligned}\right\} \tag{8.15}$$

By using these relations we will have no difficulty in constructing the phase paths. In fact, eq. (8.15) readily yields the relation

$$\frac{x^2}{C} - \frac{y^2}{Ck} = 1, \tag{8.16}$$

which is the equation of a family of hyperbolas (Fig. 47). For C = 0 we obtain two asymptotes of this family of hyperbolas, passing through the origin of coordinates:

$$\left.\begin{aligned} y &= \sqrt{k}x, \\ y &= -\sqrt{k}x. \end{aligned}\right\} \tag{8.17}$$

The origin of coordinates is the only singular point, and, except for the asymptotes, not a single integral curve passes through the origin of the coordinates. Such a singular point is called a saddle-type singular point.

On considering the direction of motion of the representative point on the phase plane, we come to the conclusion that no matter where that point may be at the initial instant (except for the origin of coordinates and the asymptotes $y = -\sqrt{k}x$) it will ultimately always move away from the origin of coordinates, and this motion will be of an aperiodic rather than an oscillatory character. The equilibrium position corresponding to a saddle-type singular point will always be unstable, in view of the fact that the motion along the asymptote $y = -\sqrt{k}x$ can never be exactly realized, since the probability of an initial state corresponding to motion toward the singular point is zero.

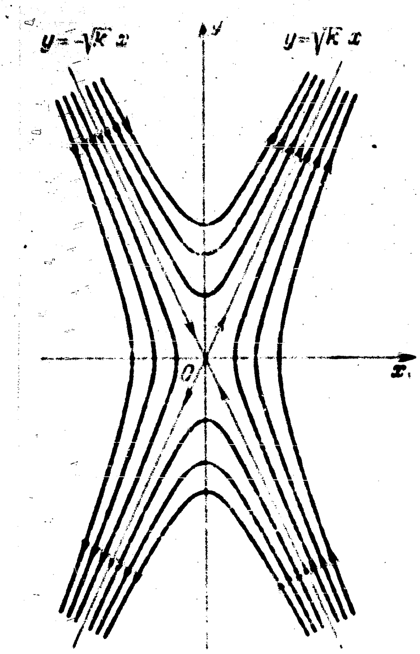

Fig. 47

Let us pass now to a consideration of the general case. First of all let us study the equilibrium points - the singular points of eq. (8.2) in which

$$P(x,\ y) = 0, \quad Q(x,\ y) = 0. \qquad (8.18)$$

Let P (x, y) and Q (x, y) be real analytic functions. Assume that the equilibrium points, i.e., the solutions of eq. (8.18), are isolated and, thus, that the number of singular points in any bounded region is finite.

Then, for the analysis of the behavior of the dynamic system in the neighborhood of a given singular point $x = x_0$, $y = y_0$, let us use

$$x = x_0 + \delta x, \quad y = y_0 + \delta y. \qquad (8.19)$$

Without disturbing the form of eq. (8.1), we may take the singular point $x = x_0$, $y = y_0$ as the origin of coordinates. Then, by substituting eq. (8.19) in eq. (8.1),

we obtain

$$\left.\begin{aligned}\frac{dx}{dt} &= cx + dy + P_2(x,\ y),\\ \frac{dy}{dt} &= ax + by + Q_2(x,\ y),\end{aligned}\right\} \tag{8.20}$$

where, for simplicity, the symbols for the variations δx, δy have been replaced by x and y and the following notation has been introduced:

$$c = P'_x(0,\ 0),\quad d = P'_y(0,\ 0),\quad a = Q'_x(0,\ 0),\quad b = Q'_y(0,\ 0),$$

while P_2 (x, y) and Q_2 (x, y) are functions having continuous partial derivatives of up to the second order inclusive and vanishing, together with their partial derivatives of the first order, at the origin of coordinates.

Neglecting, in eq.(8.20), the terms of higher order with respect to small deviations from the equilibrium point, we obtain the following system with constant coefficients:

$$\left.\begin{aligned}\frac{dx}{dt} &= cx + dy,\\ \frac{dy}{dt} &= ax + by,\end{aligned}\right\} \tag{8.21}$$

which, as is commonly known, are termed "equations of variation" about the equilibrium point.

The characteristic equation of the system (8.21) will be

$$\lambda^2 - (b + c)\lambda - (ad - bc) = 0. \tag{8.22}$$

We will consider only the cases where the characteristic roots λ_1, λ_2 are not equal to zero, so that $ad - bc \neq 0$. The corresponding critical points are termed first-order critical points, or elementary points.

The solution of the system of eq.(8.21) will be

$$\left.\begin{aligned}x &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t},\\ y &= C_1 \gamma_1 e^{\lambda_1 t} + C_2 \gamma_2 e^{\lambda_2 t},\end{aligned}\right\} \tag{8.23}$$

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where λ_1 and λ_2 are defined by the following expressions:

$$\left.\begin{aligned}\lambda_1 &= \frac{1}{2}\left[b+c+\sqrt{(b-c)^2+4ad}\right],\\ \lambda_2 &= \frac{1}{2}\left[b+c-\sqrt{(b-c)^2+4ad}\right],\end{aligned}\right\} \tag{8.24}$$

while x_1 and x_2 are the roots of the equation

$$dx^2+(b-c)x-a=0. \tag{8.25}$$

On analyzing the right-hand sides of eq.(8.24), we can easily find the ratios between the coefficients of eq.(8.21) a, b, c, d, at which the right-hand sides of eq.(8.23) will approach zero or will remain bounded as $t \to \infty$, and at which, consequently, the corresponding critical point will be stable.

For the critical point to be stable, it is necessary that $b + c < 0$ for $(b - c)^2 + 4ad \leq 0$; in the case where $(b - c)^2 + 4ad > 0$, it is also necessary, for ensuring stability, that $ad - bc < 0$, otherwise the critical point will be unstable.

If $b + c = 0$, it is necessary, for stability, that $(b - c)^2 + 4ad < 0$.

In the case where $b + c > 0$, the critical point will always be unstable.

If $ad - bc \neq 0$, the character of the critical point of eq.(8.1) will be determined essentially by the character of its first approximation (except for the case when $b + c = 0$), i.e., by the character of the solution of the system (8.21) obtained when $P(x, y)$ and $Q(x, y)$ are replaced by their terms of the first order.

It is obvious that the presence of the higher-order terms rejected by us will not modify the character of the motion in the neighborhood of the equilibrium point in the case where $\mathrm{Re}\left[\lambda_1, \lambda_2\right] \neq 0$, since it is clear that the rejected terms can only cause small additions to the "effective" values of the damping decrement. In the case where $\mathrm{Re}\left[\lambda_1\right] = 0$, or $\mathrm{Re}\left[\lambda_2\right] = 0$, the small rejected terms may affect the character of the motion, since its stability will be determined precisely by these small additions.

Consider, for example, the equation

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$$\frac{d^2x}{dt^2}+\lambda\left(\frac{dx}{dt}\right)^3+x=0, \tag{8.26}$$

which may be written in the form

$$\left.\begin{aligned}\frac{dy}{dt}&=-\lambda y^3-x,\\ \frac{dx}{dt}&=y.\end{aligned}\right\} \tag{8.27}$$

In order to study the character of the equilibrium point $x = 0$, $y = 0$, let us obtain the equations of variation

$$\left.\begin{aligned}\frac{dy}{dt}&=-x,\\ \frac{dx}{dt}&=y,\end{aligned}\right\} \tag{8.28}$$

according to which the equilibrium point $x = 0$, $y = 0$ is stable, since $b + c = 0$, $4ad = -4 < 0$.

As is clear, the character of the motion changes for eq.(8.27), since the presence of the term $\lambda\left(\frac{dx}{dt}\right)^3$ in eq.(8.26) leads to the damping of the oscillations, or to their unlimited growth, depending on whether λ is positive or negative.

We now present a theorem which is a very special case of the Lyapunov theorem.

Let, in eqs.(8.20), $P_2(x, y)$ and $Q_2(x, y)$ be functions having continuous partial derivatives up to the second order inclusive, in a certain neighborhood of the point $x = 0$, $y = 0$. Let, at this point, $P_2(x, y)$ and $Q_2(x, y)$ vanish, together with their partial derivatives of first order.

Then, if for the corresponding system of "first approximation" [eq.(8.21)], the real parts of the roots of the characteristic equation (8.22) are negative, the trivial solution $x = 0$, $y = 0$ of the system (8.20) will be stable, according to Lyapunov.

Further than that, all the solutions of the system (8.20), starting from the initial points, sufficiently close to (0, 0), will asymptotically approach a trivial

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solution as $t \to +\infty$.

The proof of this assertion may be obtained very simply as follows:

Let

$$\left.\begin{aligned} x &= U_{11}(t), \quad y = U_{21}(t), \\ x &= U_{12}(t), \quad y = U_{22}(t) \end{aligned}\right\} \tag{8.29}$$

be the solutions of the equations of first approximation for the initial conditions

$$U_{11}(0) = 1, \quad U_{21}(0) = 0,$$

$$U_{12}(0) = 0, \quad U_{22}(0) = 1.$$

It is then easy to note that equations of the type

$$\frac{dx}{dt} = cx + dy + F_1(t),$$

$$\frac{dy}{dt} = ax + by + F_2(t)$$

are integrated by means of the quadrature:

$$x = U_{11}(t)\,x(0) + \int^t \{U_{11}(t-\tau)\,F_1(\tau) + U_{12}(t-\tau)\,F_2(\tau)\}\,d\tau,$$

$$y = U_{22}(t)\,y(0) + \int^t \{U_{21}(t-\tau)\,F_1(\tau) + U_{22}(t-\tau)\,F_2(\tau)\}\,d\tau.$$

Making use of these formulas for eq. (8.20), we get

$$\left.\begin{aligned} x(t) &= U_{11}(t)\,x(0) + \int^t \{U_{11}(t-\tau)\,P_2(x(\tau),\ y(\tau)) + \\ &\quad + U_{12}(t-\tau)\,Q_2(x(\tau),\ y(\tau))\}\,d\tau, \\ y(t) &= U_{22}(t)\,y(0) + \int^t \{U_{21}(t-\tau)\,P_2(x(\tau),\ y(\tau)) + \\ &\quad + U_{22}(t-\tau)\,Q_2(x(\tau),\ y(\tau))\}\,d\tau. \end{aligned}\right\} \tag{8.30}$$

We have thereby replaced the system of differential equations (8.20) by a system of integral equations.

We emphasize that, in view of the fact that the real parts of the roots of the characteristic equation (8.22) are negative, by hypothesis, the functions $U_{ab}(t)$ will decline exponentially, since we may write

$$|U_{ab}(t)| \leqslant ke^{-\alpha t}, \tag{8.31}$$

where k and α are positive constants.

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We will not investigate the integral equations (8.30), using the conventional method of successive approximations.

Making use of the evaluation (8.31), we find that, for sufficiently small initial values x (0), y (0),

1) $|x_n(t)| \leq Ce^{-\alpha t}$, $|y_n(t)| \leq Ce^{-\alpha t}$, where C is a constant not depending on n.

2) $x_n(t)$, $y_n(t)$ converge uniformly to the solutions x (t), y (t) for all values of t over the interval $(0, \infty)$.

It follows from this that x, (t), y (t) will approach zero as $t \to + \infty$.

Let us pass now to an analysis of the character of the critical points on the phase plane.

For this purpose, we must consider the behavior of the characteristics in the neighborhood of a singular point.

The equation of the characteristics in parametric form is given by eq.(8.23). There is no difficulty in constructing from this the integral curves and in analyzing their character, depending on the relations between the coefficients a, b, c, d.

Let us consider various cases:

Let $(b - c)^2 + 4ad < 0$; in this case, the right-hand sides of eq.(8.23) have an oscillatory character. If $b + c = 0$, then in the neighborhood of a critical point the characteristics will be a family of similar ellipses surrounding the singular point which, in this case, will be a center-type singular point. If $b + c \neq 0$, then the characteristics will constitute a family of spirals, for which the asymptotic point will be a focus-type critical point, while in the case where $b + c < 0$, the characteristics approach the origin of coordinates as $t \to \infty$ so that the focus will be a stable singular point; if $b + c > 0$, then the characteristics will approach the origin of coordinates as $t \to - \infty$. In this case the focus will be an unstable singular point.

Let $(b - c)^2 + 4ad > 0$. Then, eliminating the time from eq.(8.23), we obtain the equation of the characteristics on the phase plane in the neighborhood of the

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$$\eta = \frac{1}{\lambda}\xi \ln \xi + C\xi \qquad (8.36)$$

represents the totality of curves passing through the singular point and shown in Fig. 48. The singular point is again a node, and this node will be stable at

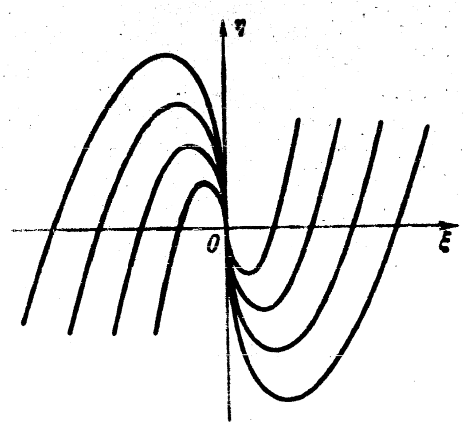

Fig. 48

$b + c < 0$ and unstable at $b + c > 0$.

The results of this consideration may be presented in the following Table:

1	$(b - c)^2 + 4ad < 0$	$b + c \neq 0$ focus $b + c = 0$ center	$b + c < 0$ stable focus $b + c > 0$ unstable focus
2	$(b - c)^2 + 4ad = 0$	node	$b + c < 0$ stable node $b + c > 0$ unstable node
3	$(b - c)^2 + 4ad > 0$	$ad - bc < 0$ node $ad - bc > 0$ saddle	$b + c < 0$ stable node $b + c > 0$ unstable node

Let us pass now from the local investigation of the character of motion near the singular points to a study of the behavior of the integral curves over the whole phase plane.

To obtain a rough idea as to the possible character of the behavior of the integral curves on the phase plane, let us first consider the approximate solution of eq. (1.1), describing the oscillations of a system with a slight nonlinearity:

$$x = a\cos\psi + \varepsilon\ldots, \quad y = \frac{dx}{dt} = -a\omega\sin\psi + \varepsilon\ldots, \qquad (8.37)$$

where the oscillation amplitude a and the phase ψ are defined by the following equations:

$$\left.\begin{aligned}\frac{da}{dt} &= -\delta(a)a,\\ \frac{d\psi}{dt} &= \omega_1(a).\end{aligned}\right\} \qquad (8.38)$$

Equations (8.37) are the equations of the phase paths in parametric form. The behavior of these phase paths will depend on the roots of the equation

$$\Phi(a) = a\delta(a) = 0, \qquad (8.39)$$

corresponding to the stationary state in the oscillatory system, and will also depend on the sign of the function $\Phi'_a(a)$.

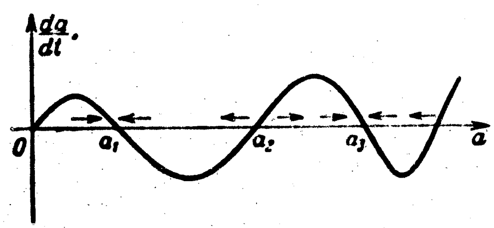

Fig. 49

The trivial root a = 0 corresponds, evidently, to the equilibrium state.

Let eq. (8.39) have a number of roots for which $\Phi'_a(a) \neq 0$. Then we obtain the position shown on graphs of the type of Fig. 49 and Fig. 50.

In the former case, however, a_1 and a_3 are roots of eq. (8.39) corresponding to

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the stable stationary state in the oscillatory system, while a_2 corresponds to the unstable state, and $a = 0$ is the unstable state of equilibrium. In the latter case, $a = 0$ will be a stable position of equilibrium, a_1 and a_3 will correspond to the unstable stationary state and a_2 to the stable stationary state. On passing to the phase plane, we obtain the point of equilibrium and a number of closed curves similar to an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{a^2\omega^2} = 1 \quad (a = a_1,\ a_2,\ a_3,\ \ldots). \tag{8.40}$$

Figures 51 and 52 give an image on the phase plane for the two types of roots of eq. (8.39) considered above.

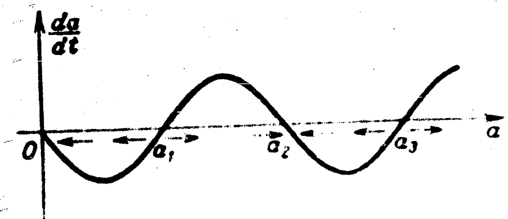

Fig. 50

On the phase plane, we have thus obtained a singular point corresponding to the 0^{th} root, and closed paths corresponding to the roots of eq. (8.39), a_1, a_2, $a_3 \ldots$ All the remaining phase paths will asymptotically approach these closed paths as $t \to \infty$ or as $t \to -\infty$.

In this case the phase plane is divided into a number of strips, wholly filled with integral curves, asymptotically approaching a certain closed integral curve which is called the limit cycle, or asymptotically approaching a point of equilibrium. The limit cycle will be stable if all the integral curves of the strip reach it as $t \to \infty$, and unstable, if they reach it as $t \to -\infty$.

We note that, in the case of a conservative oscillatory system,

$$\Phi(a) \equiv 0,$$

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and that a stationary periodic state, having an arbitrary amplitude depending only on the initial values, is possible in the system.

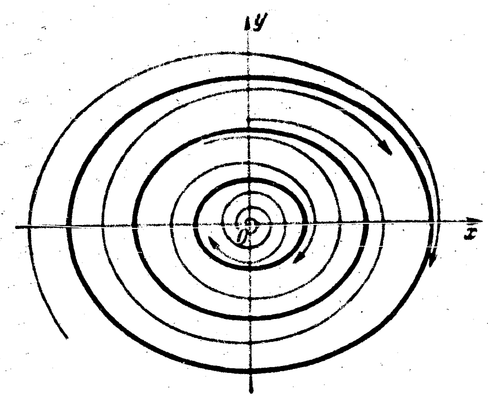

Fig. 51

This case on the phase plane corresponds to a family of closed cycles surrounding a center-type singular point (Fig. 53).

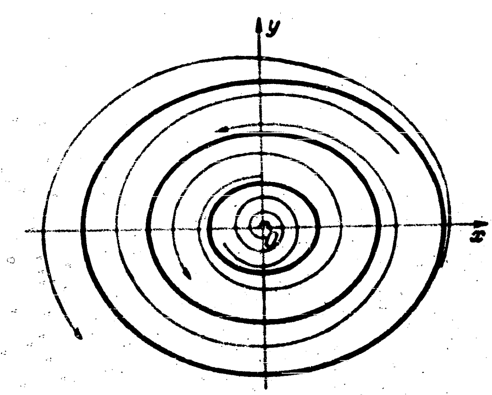

Fig. 52

Of course this reasoning is not rigorous and, what is most important, it relates

only to the special case of systems close to a linear harmonic vibrator, for which only a single singular point is possible.

The general case of the system (8.1) was investigated by Poincaré, using rigorous qualitative methods, resulting in an image of the behavior of paths on the phase plane, which is a natural generalization of that presented above.

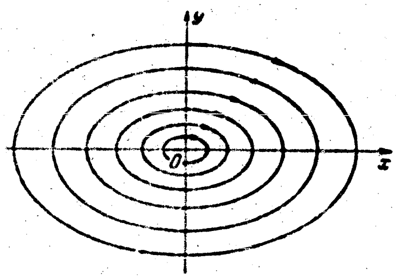

Fig. 53

Assume that the system (8.1) describes a certain oscillatory process. This means that we exclude from consideration the case when paths converging at infinity may exist on the phase plane.

Then, the phase plane will contain singular points, closed trajectories, and separatrices - which are integral curves passing through a saddle-type singular point. The separatrix plays a special role, since it divides the phase plane into a number of regions filled with paths of various types (Fig. 54).

Any unclosed integral curve either winds toward a limit cycle, or approaches one or a number of singular points.

Closed cycles may form either a continuous or a discrete family.

The image for which the cycles form a continuous family enclosing a center-type singularity is, from the physical point of view, typical for conservative systems.

In the case where the cycles form a discrete family they are within the strips, and all the integral curves included in such a strip will asymptotically approach a cycle which, in this case, is the limit cycle. If the approach to this cycle is

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made as $t \to \infty$, we speak of a stable limit cycle. If this approach takes place as $t \to -\infty$, we naturally speak of an unstable cycle.

Such a phase image is characteristic for self-sustained oscillatory systems. It is obvious that the stable limit cycles form stationary states of self-sustained oscillations, and that the stationary state in a self-sustained oscillatory system is independent of the initial conditions lying within certain limits. Of course in

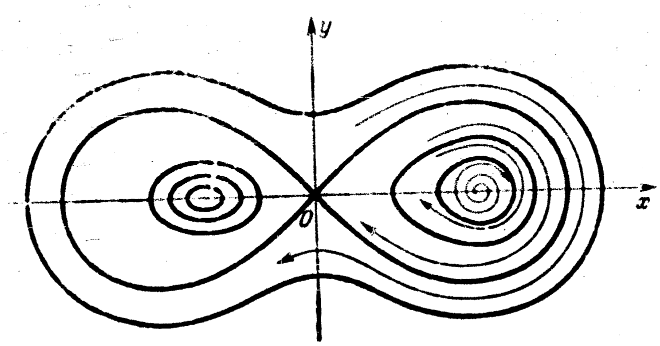

Fig. 54

the case of the presence of several limit cycles, a jump from one limit cycle to another is possible at any change in the initial conditions.

This picture of the behavior of paths on the phase plane is very clear, but is amenable to more detailed treatment only with great difficulty, in the solution of concrete examples.

Up to now there have been no sufficiently general theoretical methods for solving the question of the existence of limit cycles and of determining their location, except for the case of systems close to linear ($\varepsilon \ll 1$)

In the investigation of problems of this kind the concept of the index, as introduced by Poincaré, is often of great importance.

Let us take, on the phase plane under consideration, a certain closed curve Γ,

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and assume that this curve is simple (i.e., that it has no double points) and does not pass through a state of equilibrium. Let us take, on this curve, a certain point S and lay through it a vector coinciding in direction with the tangent to the phase path passing through this point.

Let the continuous function $\theta(t)$ determine the angle formed by this vector with the positive direction of the axis Ox.

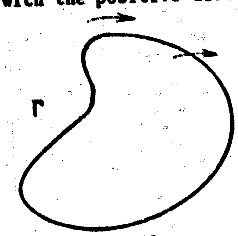

Fig. 55

If the point S makes a full revolution along the closed curve Γ and returns to its original locus, the vector will make a certain number of revolutions during this period; consequently, $\theta(t)$ will vary by the quantity $2\pi j$, where j is a positive or negative integer (Fig. 55). The number j is called the index of the closed contour Γ.

It is obvious that the number j may be expressed by the curvilinear integral

$$j=\frac{1}{2\pi}\oint_{\Gamma} d\left(\operatorname{arctg}\frac{dy}{dx}\right)=\frac{1}{2\pi}\oint_{\Gamma}\frac{P\,dQ-Q\,dP}{P^2+Q^2}. \quad (8.41)$$

This curvilinear integral is taken from the total differential; consequently, if the integrand function and its derivatives are continuous within the region encompassed by the curve Γ, the integral will vanish.

It is entirely clear that such properties of continuity can be violated only at points for which

$$P^2+Q^2=0,$$

i.e., at the singular points of our equation.

For this reason, if no singular points exist within the curve Γ, its index will be equal to zero.

In the general case, when there are a number of singular points inside Γ, we perform the construction shown in Fig. 56.

In this case the sum of the integrals of eq. (8.41), taken along the curve Γ and the curves Γ_1, Γ_2, Γ_3, ..., surrounding the singular points, vanishes, since no singular points exist in the corresponding multiply-connected region bounded by the

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curves Γ, Γ_1, Γ_2, Γ_3, ...

Since a circuit in a direction opposite to that adopted in the definition of the index is attributed to the curves Γ_1, Γ_2, Γ_3 ..., it is obvious that the index of the curve Γ is the sum of numbers depending only on the properties of the corresponding singular point.

These numbers are known as indexes of singular points. For a given determination, the index of any singular point is equal to the integral

$$\frac{1}{2\pi}\oint_{\Gamma}\frac{P\,dQ - Q\,dP}{P^2+Q^2},$$

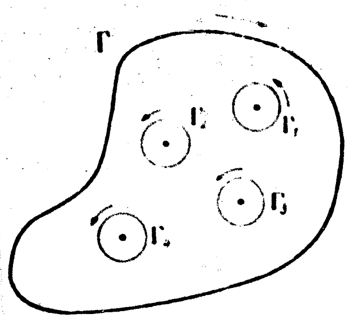

Fig. 56

taken along the curve Γ, encircling only one given point M, and not depending on the form of Γ.

For the actual determination of the index, it is consequently possible to take, as Γ, the ellipse

$$(ax+by)^2+(cx+dy)^2=1, \quad (8.42)$$

which is as close as may be desired to the point of equilibrium. Obviously, here the origin of coordinates was placed at the singular point under consideration.

Then, neglecting the terms of a higher order of smallness in the expressions for P (x, y) and Q (x, y), we have

$$j=\frac{q}{2\pi}\oint(x\,dy-y\,dx), \quad (8.43)$$

where

$$q=ad-bc,^{*}$$

or, by virtue of the well-known expression for the area in terms of the curvilinear integral,

$$j=\frac{q}{\pi}S,$$

* Translator's note: See errata sheet.

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where $S = \frac{\pi}{|q|}$ is the area of the ellipse, so that

$$I = \frac{q}{|q|}.$$

It directly follows from this that the Poincaré index for a node, a focus, or a center is equal to $+1$, and for a saddle, to -1. These same results could be obtained by starting from a direct consideration of the image on the phase plane see (Fig. 57).

It must be emphasized that, even if the character and location of the singular points are known, this is still entirely inadequate for forming a general idea on the behavior of the integral curves of the differential equations (8.2). For a number of special cases, however, we are able to reach important conclusions.

Consider, for instance, a certain closed integral curve defined by eq.(8.2), without double or singular points.

Since the vector tangent to this curve, on making a complete circuit of the curve in a positive direction will rotate through the angle 2π, the sum of the indices of all the singular points inside the closed integral curve will consequently be equal to 1.

It is thus evident that, inside a closed integral curve, there must be at least one singular point and that, if only one such point is present, it must be a center, a focus, or a node.

If the closed integral curve encloses several singular points, then the number of saddle-type singular points per unit is less than the number of singular points of the other types.

The concept of index makes it possible to determine the number and position of limit cycles for a given equation, i.e., to isolate, on the phase plane, a certain number of rings of finite width, each containing only one limit cycle.

For this purpose it is necessary to pick out several singular points whose index sum is equal to $+1$, and to surround them by two closed curves in such a way

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that no singular points remain in the ring-shaped region so obtained.

After this, we must investigate the direction of the velocity vector of the representative point on these curves. Depending on the direction of this vector, conclusions as to the existence of a limit cycle, and also as to its character, can

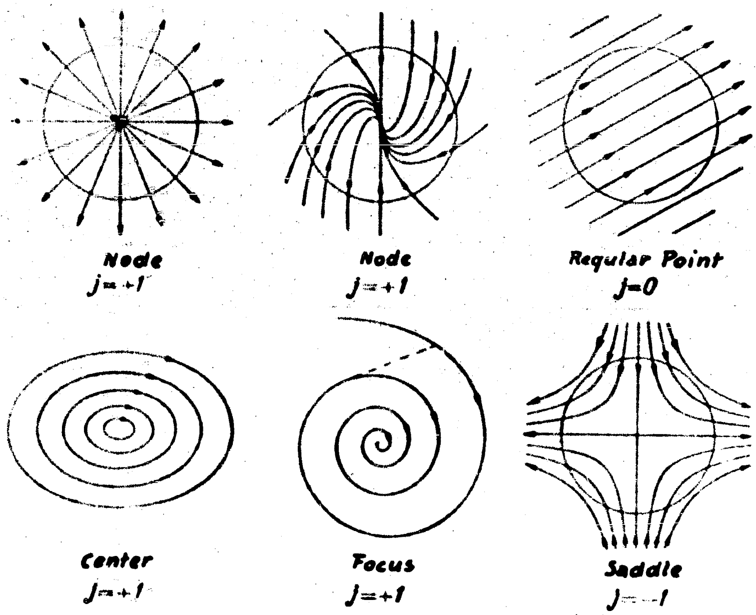

Fig. 57

be drawn. For example, if the velocity vector of the representative point everywhere tends inward toward the ring-shaped region, at least one stable limit cycle will exist in this region. If the velocity vector everywhere tends outward, at least one unstable limiting cycle is present, etc.

We present in conclusion, an elegant example (Bibl. 49) for which it is easy to determine the limit cycle analytically.

Consider the equation

$$\frac{dy}{dx} = \frac{(x+y)\sqrt{x^2+y^2}-y}{(x-y)\sqrt{x^2+y^2}-x}. \qquad (8.44)$$

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On substituting the variables $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, where ρ and φ are polar coordinates, we obtain the equation

$$\frac{d\rho}{d\varphi} = \rho - 1, \tag{8.45}$$

when total integral will be

$$\rho = 1 + Ce^{\varphi}, \tag{8.46}$$

where C is an arbitrary constant. For ρ to be always positive, it is necessary that $\varphi \leq \ln |C|$ for $C < 0$. The family of integral curves, in this case, will consist of a circle with a radius of $\rho = 1$ (for $C = 0$) and of spirals starting at the origin of coordinates and asymptotically approaching the circle from within (developing along the circle from within) while $\varphi \to -\infty$ (for $C < 0$) and asymptotically approaching the circle from without (winding along the circle from without) while $\varphi \to -\infty$ (for $C > 0$). The circle with the radius $\rho = 1$ will, in this case, represent the limit cycle of eq. (8.44); see (Fig. 58).

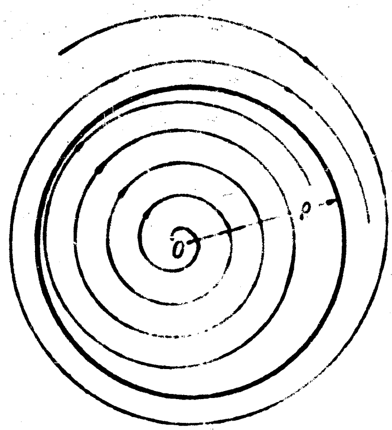

Fig. 58

Section 9. The Liénard Method

In many important special cases, it is convenient to investigate the nonlinear differential equation (8.2) by means of a graphical construction of integral curves on the phase plane. A very elegant method of graphic construction of the integral curves is the method proposed by the French engineer Liénard (Bibl. 24). By this method, all types of motion permitted by the given equation can be studied and the limit cycles found.

Liénard investigated an equation of the form

$$\frac{d^2x}{dt^2}+f(x)\frac{dx}{dt}+g(x)=0. \tag{9.1}$$

Equations of this type, as is commonly known, include the van der Pol equation, the Rayleigh equations, and others that could be cited.

After Liénard, the problem of establishing the existence criteria and the uniqueness of the limit cycle for equations of the type of eq.(9.1) were investigated by several authors. We may mention, for example, the work of V.S.Ivanov, of Levinson and Smith, and of A.V.Dragilev.

A formulation of A.V.Dragilev's theorem is presented here. We introduce the notation

$$F(x)=\int_0^x f(x)\,dx, \quad G(x)=\int_0^x g(x)\,dx. \tag{9.2}$$

Then,

1) if g (x) satisfies the Lipschitz condition

$$xg(x)>0, \quad x\neq 0; \quad G(\infty)=\infty;$$

2) if F (x) is uniquely determined in the interval $\infty < x < \infty$ and, for each finite interval, satisfies the Lipschitz condition, while, for sufficiently small $|x|$ $F(x) < 0$ at $x > 0$, and $F(x) > 0$ at $x < 0$;

3) if a number M and numbers k and k', k' < k are in existence, so that

$$F(x)\geqslant k, \quad \text{at} \quad x>M,$$
$$F(x)\leqslant k', \quad \text{at} \quad x<-M,$$

then eq.(9.1) has at least one limit cycle.

It is clear that, under very general conditions, the existence of at least one limit cycle is thus established.

The question of the uniqueness of the limit cycle is the subject of the Levinson and Smith theorem.

Let:

1. g (x) be an odd function so that g (x) > 0, for x > 0;

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2. $F(x)$ be an odd function with a value x_0 so that $F(x) < 0$ for $0 < x < x_0$, and $F(x) \geq 0$ and increases monotonously, for $x \geq x_0$;
3. $F(\infty) = G(\infty) = \infty$;
4. $f(x)$ and $g(x)$ satisfy the Lipschitz conditions over any finite interval.

In this case, eq.(9.1) has a limit cycle and, at that, a unique limit cycle.

No proof for these theorems will be given here*.

Consider the simpler case when the following limiting conditions are satisfied:

1. $f(x)$ is an even function, $g(x)$ is an odd function and, in addition, $xg(x) > 0$ for any values of x, while $f(0) < 0$;
2. $f(x)$ and $g(x)$ are analytic functions;
3. $F(x) \to \infty$ while $x \to \infty$;
4. The equation $F(x) = 0$ has a single positive root $x = a$ and, in addition, for $x \geq a$, the function $F(x)$ increases monotonously.

As readily demonstrated, these conditions are satisfied by the van der Pol equation and also by the Rayleigh equation.

We will prove that, when the above conditions are satisfied, eq.(9.1) has a unique closed cycle which will be stable. The proof will be furnished by means of the very clear and elementary method given in the book by Lefschetz (Bibl.23). We put

$$y = \frac{dx}{dt} + F(x), \quad \lambda(x, y) = \frac{y^2}{2} + G(x). \tag{9.3}$$

With this notation, $\frac{y^2}{2}$ may be interpreted as the kinetic energy, and the above-mentioned function $G(x)$ may be interpreted as the potential energy.

Let us now determine the energy dissipated by the system under the oscillations defined by eq.(9.1). We have

$$\frac{d}{dt}\left(\frac{y^2}{2} + G(x)\right) = \frac{d}{dt}\left\{\frac{1}{2}\left(\frac{dx}{dt} + F(x)\right)^2 + G(x)\right\} =$$
$$= \frac{dx}{dt}\left(\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x)\right) + F(x)\frac{d}{dt}\left(\frac{dx}{dt} + F(x)\right),$$

* The proof for these theorems is given by V.V.Nemytskiy and V.V.Stepanov (Bibl.31).

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or, taking eq.(9.1) and the notation of eqs.(9.2) and (9.3) into consideration, we find, after canceling out dt,

$$d\lambda = F(x)\,dy. \qquad (9.4)$$

In this way the energy dissipated by the system will be expressed by the magnitude of the integral $\int F(x)\,dy$ taken along the integral curve.

Passing to the variables x, y, eq.(9.1) will yield the equivalent system

$$\left.\begin{aligned} \frac{dx}{dt} &= y - F(x), \\ \frac{dy}{dt} &= -\,g(x). \end{aligned}\right\} \qquad (9.5)$$

Thus we have to prove that the system (9.5) has a unique and stable cycle.

The system of eq.(9.5) has the following obvious properties:

1) if x (t) and y (t) are solutions of the system of equations (9.5), then, by virtue of the imposed limitations, $-x(t)$, $-y(t)$ will likewise be solutions (since F (x) is an odd function); consequently, the curve symmetric to an integral curve with respect to the origin of coordinates will also be an integral curve of eq.(9.6);

2) the single critical point of the system (9.5) on the phase plane is the origin of coordinates, and therefore the limit cycle must be circumscribed about origin of coordinates;

3) the slope of the integral curve Γ is determined by the following equation:

$$\frac{dy}{dx} = -\,\frac{g(x)}{y - F(x)}. \qquad (9.6)$$

Since g (0) = 0, all the tangents to the path Γ at points lying on the axis Oy (except the origin of coordinates), are horizontal.

On the other hand, if we consider the curve Δ, whose equation will be $y - F(x) = 0$ (Fig.59, broken line), then it is not hard to see that all tangents to Γ at the points of its intersection with Δ are vertical except at the origin of coordinates (since on Δ, $y - F(x) = 0$ and, consequently, $\frac{dy}{dx} = \infty$). Moreover, since

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$g(x)$ is odd, it follows that $xg(x) > 0$; then, according to eq.(9.5), y decreases along the curve Γ to the right of the axis Oy and increases to the left of the axis Oy. In addition, x increases when Γ lies above Δ [since, in this case, $y - F(x) > 0$], and decreases, when Γ lies below Δ. Consequently, the curve Γ has the form shown in Fig.59.

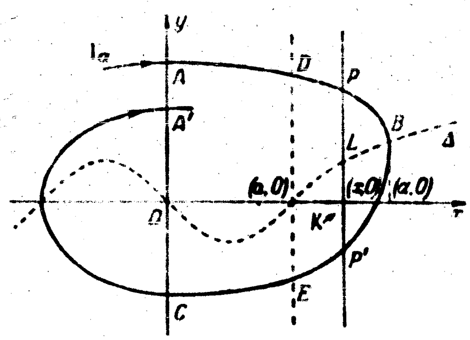

Fig.59

Let α denote the abscissa of point B, and let Γ_α replace Γ.

We will now establish the conditions under which Γ_α will be a closed cycle.

It is obviously necessary that $OA' = OA$, since, if this is not the case a repetition of our argument will show that a prolongation of Γ beyond the point A', because of the fact that the cycle cannot intersect itself, will give the point A'' lying below A' (Fig.60), and so on. In this way, if $OA' \neq OA$, the curve Γ_α cannot return either to the point A nor to the point A' and, consequently, cannot be closed. Hence, Γ_α must intersect each axis at two and two points only. It follows from this that $OA = -OC$.

Now assume that $OA \neq -OC$, and let the points A' and C' be symmetric to the points A and C with respect to the origin of coordinates. According to the first property of the system of eq.(9.5), a curve symmetric to the curve Γ_α with respect

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to the origin of coordinates will be the closed integral curve Γ_1, passing through the points A', C'. Since the axis Oy is perpendicular to Γ_α, we arrive at the position shown in Fig.61 where the curves Γ_α and Γ_1 intersect, which is impossible. Therefore OA = – OC.

On the other hand, assume that OA = – OC. Then a curve symmetric to the arc AC with respect to the origin of the

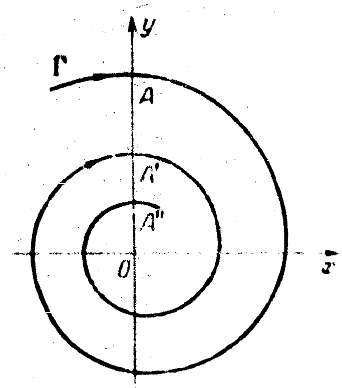

Fig.60

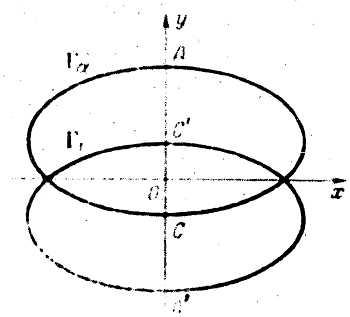

Fig.61

coordinates will be an arc of the cycle joining the point A to the point C at the left of the axis Oy. Together with the arc AC, this will form a closed cycle.

Thus, for Γ_α to be a closed cycle, it is necessary and sufficient that OA = – OC.

Since, according to the symbols in eq.(9.2), $\lambda\,(0,\ y) = \frac{y^2}{2}$, the latter condition may be formulated in the following way:

For Γ_α to be a closed cycle, it is necessary and sufficient that

$$\lambda(A) = \lambda(C). \qquad (9.7)$$

We can show that, if the conditions satisfied by the functions f (x) and g (x) are met, eq.(9.7) will be valid so that eq.(9.1) will have a limit cycle.

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As a proof let us consider the following curvilinear integrals taken along the curve Γ.

Let

$$\varphi(\alpha) = \lambda(C) - \lambda(A) = \int\limits_{ABC} d\lambda = \int\limits_{ABC} F(x)\, dy. \tag{9.8}$$

If $\alpha \leq a$ (cf. Fig.59), then $dy < 0$ and, according to the fourth condition (cf. above), also $F(x) < 0$; in this way, $\varphi(\alpha) > 0$, i.e., $\lambda(C) > \lambda(A)$.

Consequently, Γ_α cannot be a closed cycle. (In this case, $\int\limits_{ABC} F(x)\, dy > 0$, i.e., the energy dissipated by the system is positive and, obviously, no undamped oscillations can exist in the system.)

For this reason, assume that $\alpha \geq a$, i.e., that the curve Γ_α has the form shown in Fig.59. Let us denote

$$\varphi_1(\alpha) = \int\limits_{AD} d\lambda + \int\limits_{CE} d\lambda, \quad \varphi_2(\alpha) = \int\limits_{DBE} d\lambda;$$

Then,

$$\varphi(\alpha) = \varphi_1(\alpha) + \varphi_2(\alpha).$$

On the basis of eqs.(9.4) and (9.6) we may write

$$d\lambda = F(x)\frac{dy}{dx}\, dx = -\frac{F(x)\, g(x)}{y - F(x)}\, dx. \tag{9.9}$$

Since $F(x) < 0$ for $x < a$, then $d\lambda$ is positive when Γ_α is described in the direction from A to D or from E to C so that $\varphi_1(\alpha) > 0$. On the other hand, along DBE we have $d\lambda < 0$ and, consequently, $\varphi_2(\alpha) < 0$.

It is obvious that when α increases the arc AD will rise, while the arc CE will sink so that, for a fixed x, $|y|$ will increase. Since, for $\varphi_1(\alpha)$, the limits of integration, bearing in mind eq.(9.9), are fixed (from $x = 0$ to $x = a$), then an increase in α will cause a decrease in $\varphi_1(\alpha)$ since $d\lambda = \dfrac{g(x)}{\dfrac{y}{|F(x)|} \pm 1}\, dx$ decreases with increasing y.

Let us pass now to an evaluation of the character of the variation of $\alpha_2(\alpha)$ with increasing α. Let α_1 and α_2 be two successive values of α, and let $\alpha_2 > \alpha_1$. We can show that

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$$\varphi_2(\alpha_2) < \varphi_2(\alpha_1).$$

Drop the perpendiculars D_1D_1' and E_1E_1' to the straight line D_2E_2 (Fig.62). Then,

$$\int\limits_{D_2B_2E_2} F(x)\,dy = \int\limits_{D_2D_1'} F(x)\,dy + \int\limits_{D_1'E_1'} F(x)\,dy + \int\limits_{E_1'E_2} F(x)\,dy. \qquad (9.10)$$

Since, in this case, $F(x) > 0$ and $dy < 0$, it follows that

$$\int\limits_{D_2B_2E_2} F(x)\,dy < \int\limits_{D_1'E_1'} F(x)\,dy. \qquad (9.11)$$

From the very construction of D_1' and E_1' we see that y varies along the curves $D_1B_1E_1$ and $D_1'E_1'$ within the same limits (from a higher to a lower value).

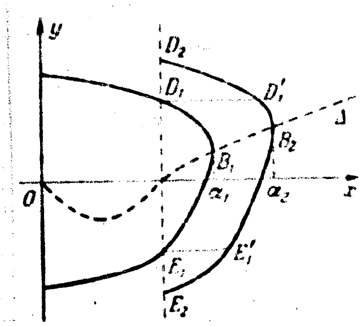

Fig.62

On the other hand, for a given y, the abscissa x of the point of the curve $D_1'E_1'$ will be greater than for the corresponding point of the curve $D_1B_1E_1$.

For this reason, for a given y, F (x) on $D_1B_1E_1$ will be smaller than F (x) on $D_1'B_1'$. Consequently, since $dy < 0$,

$$\int\limits_{D_1'E_1'} F(x)\,dy < \int\limits_{D_1B_1E_1} F(x)\,dy, \qquad (9.12)$$

and, from eq.(9.11) we find

$$\int\limits_{D_2B_2E_2} F(x)\,dy < \int\limits_{D_1B_1E_1} F(x)\,dy; \qquad (9.13)$$

i.e., actually $\varphi_2(\alpha_2) < \varphi_2(\alpha_1)$ for $\alpha_2 > \alpha_1$.

Thus, $\varphi(\alpha) = \varphi(\alpha) + \varphi_2(\alpha)$ for $\alpha \geq 0$ is a monotonously decreasing function of α.

We note that, in the case $\alpha \leq a$, we have

$$\varphi(\alpha) = \varphi_1(\alpha) > 0.$$

We will show that

$$-\varphi_2(\alpha) \to \infty \quad \textit{for} \quad \alpha \to \infty.$$

For this purpose, we establish some value of x_1 such that

$$a < x_1 < \alpha,$$

and lay the axis PP' parallel to the axis Oy through the point x_1 on the axis Ox (see Fig. 59).

We then have

$$\int_{DBE} d\lambda < \int_{PBP_1} d\lambda = \int_{PBP_1} F(x)\,dy. \qquad (9.14)$$

However, for the arc PBP_1 we have $x \geq x_1$ and, consequently,

$$F(x) \geqslant F(x_1).$$

We, therefore, find that

$$\int_{DBE} d\lambda \leqslant F(x_1)\int_{PBP_1} dy = -F(x_1)|\overline{PP_1}|,$$

whence

$$-\varphi_2(\alpha) = -\int_{DBE} d\lambda > \overline{KL}\cdot\overline{KP}. \qquad (9.15)$$

It is clear that the segments $\overline{KP}$ and $\overline{KL}$ may be taken as large as desired for sufficiently large values of α.

Thus, in reality,

$$-\varphi_2(\alpha) \to \infty \quad \text{as} \quad \alpha \to \infty.$$

In this way we have shown that $\varphi(\alpha)$ is a function monotonously decreasing from the values of $\varphi(\alpha) > 0$ to $\varphi(\alpha) = -\infty$ while $\alpha \to \infty$. Consequently, $\varphi(\alpha)$ vanishes once and once only for $\alpha = \alpha_0$, and Γ_{α_0} will be the required unique closed characteristic since, for it, the condition (9.7) will be satisfied.

We will show that Γ_{α_0} is a stable limit cycle.

If $\alpha < \alpha_0$, then $\varphi(\alpha) > 0$ and, consequently, $\lambda(C) > \lambda(A)$.

If $\alpha > \alpha_0$, then $\varphi(\alpha) < 0$, and, consequently, $\lambda(C) < \lambda(A)$.

Let the points A_0 and C_0 correspond to the intersection of Γ_{α_0} with the y axis; then, as is obvious, the point C is closer to Γ_{α_0} than the point A if $\alpha < \alpha_0$, so that the point A' is closer to Γ_{α_0} than A.

By analogous reasoning for the case $\alpha > \alpha_0$, we arrive at the conclusion that the limit cycle Γ_{α_0} is stable.

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We will now discuss the method of actual construction of the integral curves on the phase plane.

Liénard's graphic method is generally used where the elastic force g (x) is linear with respect to x. In this case, by an appropriate selection of the new variables, we may, without interfering with the generality, reduce eq.(9.6) to the form

$$\frac{dy}{dx}=-\frac{x}{y-F(x)}. \qquad (9.16)$$

The method of graphic integration of equations of the type of eq.(9.16),

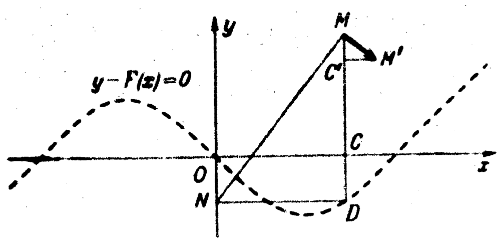

Fig. 63

proposed by Liénard, is as follows: On the phase plane, let us construct the curve Δ, whose equation is (Fig. 63)

$$y-F(x)=0 \qquad (9.17)$$

After constructing this curve, the direction of the tangent to the integral curve of eq.(9.16), passing through any point of the phase plane, can be graphically determined: From the point M (x, y), for which we seek the direction of the tangent, let us drop the perpendicular to the abscissa $\overline{MO}$ and extend it to its intersection with the curve Δ at the point D. From the point D let us drop a perpendicular to the ordinate $\overline{DN}$. Then, the line $\overline{NA}$ will be perpendicular to the integral curve of eq.(9.16) passing through the point M. If the phase point of eq.(9.16), at the time t = 0, coincides with the point M (x, y), it will be dis-

placed (after the time segment dt) along the ordinate by the segment

$$dy = -x\,dt = \overline{ND}\,dt = \overline{MC'},$$

and along the abscissa, by the segment

$$dx = (y - F(x))\,dt = \overline{MD}\,dt = \overline{C'M'}.$$

Since the triangles NDM and MC'M' are similar, it follows that

$$\frac{\overline{MC'}}{\overline{ND}} = \frac{\overline{C'M'}}{\overline{DM}} = \frac{\overline{M'M}}{MN}$$

and, consequently, $\overline{M'M} \perp \overline{MN}$.

Thus, in order to produce, to a given point M (x, y) in the x, y plane, the tangent to the integral curve Γ, passing through this point M (x, y), it is sufficient to produce the vertical straight line MCD and the horizontal straight line DN, and to join the points N and M. The required tangent to the curve Γ will be perpendicular to the straight line NM, whence it follows that, having the arbitrary curve Δ and the arbitrary initial conditions x_0, y_0 mapped by the point M_0 (x_0, y_0), it is easy to find the direction of the tangents, and, consequently, to construct the approximate integral curve.

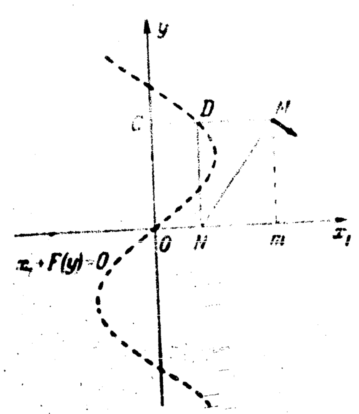

Fig. 64

Thus, to construct the integral curve Γ, passing through an assigned point of the phase plane M (x, y), we proceed as follows: From the above-described construction we find the tangent to the given point and replace the integral curve in the neighborhood of this point by a small segment of the tangent. Then, at the end of the resultant segment, we again determine the direction of the tangent, and in the neighborhood of the new point we replace the integral curve by a segment of a straight-line, giving the approximate integral curve in the form of

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a broken line. The degree of accuracy will depend on the magnitude of the individual links.

For the transformation of eq. (9.1) it is in many cases convenient, not to substitute the variables by eq. (9.3) but to perform the substitution by the formula

$$x_1 = \int x\,dt$$

and to consider the equation in the form

$$\frac{d^2x_1}{dt^2} + F\left(\frac{dx_1}{dt}\right) + x_1 = 0 \qquad (9.18)$$

or, using the notation $\frac{dx_1}{dt} = y$ and eliminating the time t,

$$y\frac{dy}{dx_1} + F(y) + x_1 = 0. \qquad (9.19)$$

In this case, the equation of the auxiliary curve will be

$$x_1 + F(y) = 0, \qquad (9.20)$$

so that, on the phase plane, we obtain the construction shown in Fig.64. Dropping perpendiculars from the point M to the abscissa $\overline{Mm}$ and to the ordinate $\overline{MC}$, and also dropping a perpendicular from the point D to the abscissa, eq. (9.19) will yield

$$\overline{Nm} = -y\frac{dy}{dx_1}. \qquad (9.21)$$

Consequently, eq. (9.19) may be written in the form

$$\overline{Nm} = \overline{CM} - \overline{CD}, \qquad (9.22)$$

since $\overline{CM} = x_1$, $\overline{CD} = -F(y)$*.

Thus in our case, as well, we may perform the construction of approximate integral curves according to the above scheme.

If the curve Δ is symmetric with respect to the origin of coordinates, then the integral curves Γ so constructed will roll along closed curves - limit cycles - corresponding to the periodic state whose existence and stability have been proved above.

* Translator's note: See errata sheet.

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We note that the graphic construction, proposed by Liénard, does not postulate that the curve Δ must necessarily be symmetric. This graphic method is also applicable to the case where Δ is more or less close to a symmetric curve, applicable, for example, to the curve determined by the characteristic of a neon tube, etc. In this case, the curve Δ need not necessarily be represented by an algebraic equation and may, instead, be obtained experimentally. This feature is very important from the practical point of view.

Below, we present a few examples to illustrate the above-described graphic construction of integral curves.

We note that, for certain special cases, the Liénard construction immediately yields the integral curve, eliminating the necessity of constructing an approximate broken line.

For example, in the case of free linear oscillations described by the equation

$$\frac{d^2x}{dt^2} + x = 0, \tag{9.23}$$

the equation of the phase paths will be

$$\frac{dy}{dx} = -\frac{x}{y}. \tag{9.24}$$

In this case, the equation of the curve Δ will be $x = 0$; and the point N will coincide with the origin of coordinates for all assigned values of the point D. Consequently, the integral curves will be circles having their center at the origin of coordinates.

If the oscillations of the system take place under the influence of a linear elastic force in the presence of Coulomb friction, the equation of motion may be presented in the form

$$\frac{d^2x}{dt^2} + A\,\mathrm{sign}\,x + x = 0. \tag{9.25}$$

In this case, we obtain the following equation for the curve Δ:

$$x = A \quad \text{at} \quad x > 0,$$

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$$x = -A \quad \text{as} \quad x < 0. \tag{9.26}$$

Obviously, for the integral curve Γ, the point N will coincide in the upper half-plane with the point S_1, and in the lower half-plane with the point S_2 (Fig.65), regardless of the assigned values of the point D. In this way, the integral curve Γ will consist of arcs of circles with their centers at the points S_1 and S_2. These arcs will merge at the intersection of the integral curve with the axis Ox. In this case, it is obvious that the amplitude of the damped oscillations will decrease by the quantity 2A with each passage between two successive rest positions of y = 0, until the oscillatory system finally reaches a state of rest.

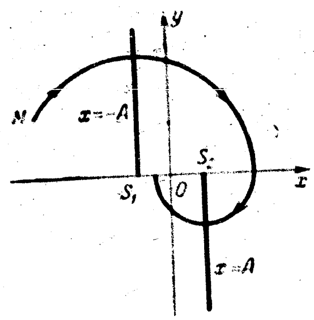

Fig.65

Let us now integrate the van der Pol equation by the Liénard method, taking it in the form

$$\frac{d^2x}{dt^2} - \varepsilon\left(1 - \left(\frac{dx}{dt}\right)^2\right)\frac{dx}{dt} + x = 0.$$

The equation of the curve Δ on the phase plane will be

$$x + \varepsilon(1 - y^2)y = 0, \tag{9.28}$$

where ε is a certain parameter.

The properties of the function $-\varepsilon(1 - y^2)y$ are as follows:

1) at y = 0 and y = ± 1, x = 0;

2) at $y = \pm\frac{1}{\sqrt{3}}$, x will assume the extreme values (Fig.66)

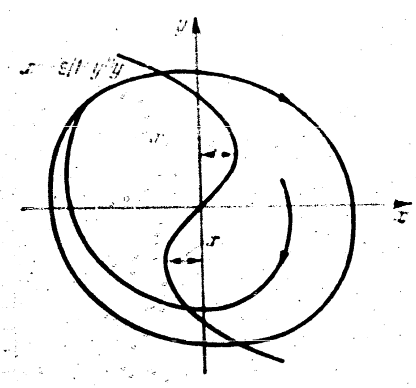

Fig.66

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$$x = \pm \frac{2\varepsilon}{\sqrt{3}}$$

According to this, an increase in ε causes the loop to elongate along the x axis and to approach the pair of straight lines $y = \pm 1$ (Fig.67).

In the case where $\varepsilon = 0$, the integral curves of the equation

$$\frac{dy}{dx} = \frac{-\varepsilon(1 - y^2)y - x}{y} \tag{9.29}$$

form a family of concentric circles with the center at the origin of coordinates; then, this equation will correspond to simple harmonic oscillations.

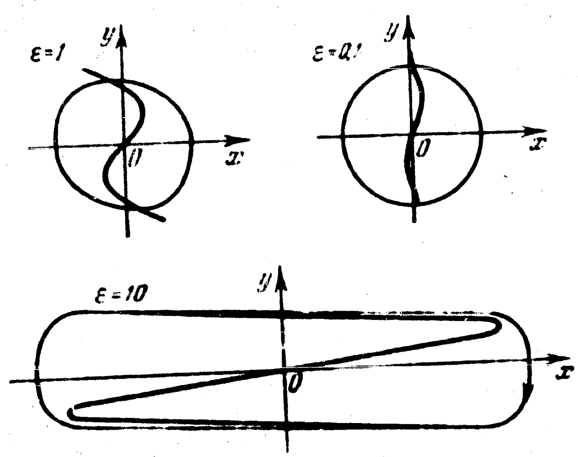

Fig.67

For $\varepsilon \neq 0$, we will study the behavior of the integral curves of eq.(9.27) by means of Liénard's graphic method. According to this method, let us construct the field of directions for the curve of eq.(9.28) and let us find the limit cycles. Figure 67 gives the curves of eq.(9.28) constructed for three different values $\varepsilon = 0.1$, $\varepsilon = 1$, and $\varepsilon = 10$, respectively. In these same graphs, the limit cycles of

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the system have been constructed by the Liénard method. Since, as generally known, $\varepsilon > 0$ in the case under consideration, the origin of coordinates is an unstable equilibrium position so that all the integral curves leaving the origin of coordinates will describe expanding spirals about it. However, the spirals unwinding about the origin of coordinates cannot extend into an indefinite distance, since, for

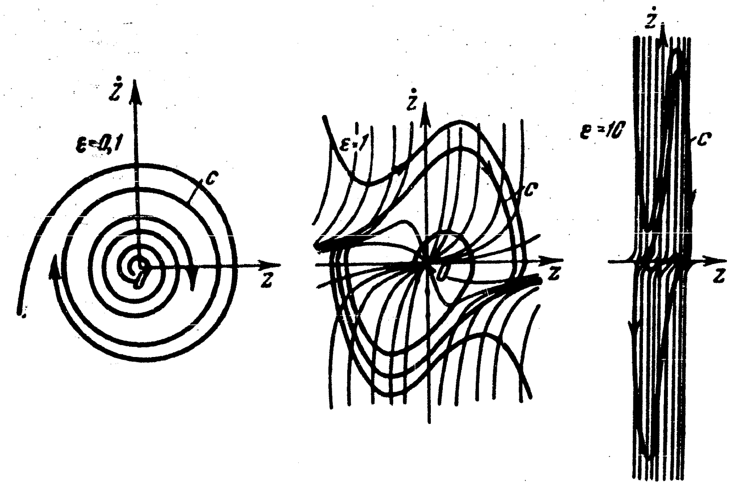

Fig.68

great values of y, the damping in the oscillatory system described by eq.(9.27) becomes positive.

As each spiral expands, its successive loops approach more and more, and all spirals asymptotically wind from within toward a closed curve, the limit cycle.

Along this limit cycle, spirals close to the origin of coordinates and spirals far from the origin will wind. The closed integral curve, the limit cycle toward which all integral curves of eq.(9.29) tend, corresponds to the periodic solution of eq.(9.27).

We note that the closed cycle contains one singularity with the index +1, that

for $\varepsilon = 0.1$ and $\varepsilon = 1.0$ this point will be an unstable focus, and that for $\varepsilon = 10$, we have an unstable node.

On the basis of Fig.67 the variation in the character of the motion within the system with any variation in the parameter ε can be estimated. For any values of ε in the system, self-sustained oscillations take place, but the width and form of these self-sustained oscillations and the character of their build-up differ. For comparison, Fig.68 shows the results of the numerical integration of eq.(9.29) at

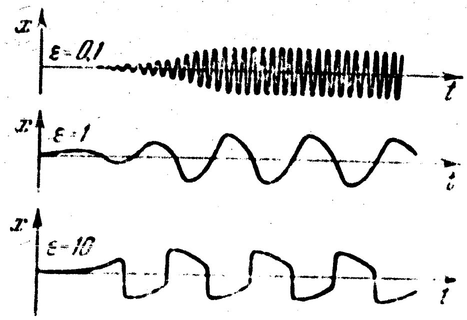

Fig.69

the same values of the parameter ε, as well as the curves characterizing the variation of x with time (Fig.69).

In conclusion we note that Rensuki Usui (Bibl.30), combining the Liénard method (developed in detail by the author for the case of a symmetric characteristic) and the Kirstein method (which is exceedingly difficult for practical application), has developed a standard graphic method for solving the nonlinear differential equations that describe the processes in self-excited systems. The method developed by him may be used for the consideration of the oscillatory processes in complex circuits and also in connected circuits.

This method will not be further discussed here, and the interested reader is

referred to special literature on this subject.

Section 10. Relaxation Oscillatory Systems

Up to now we have considered the van der Pol equation mainly for small values of ε, and only in Section 8 have we pointed out the changes that take place in the solution as ε increases.

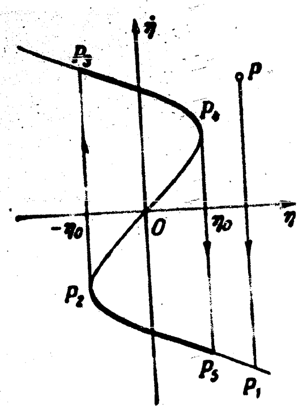

Fig. 70

Let us now consider the van der Pol equation at large values of ε, and, in particular, let us try to find the asymptotic form of the solution as $\varepsilon \to \infty$.

For this investigation, it is convenient to take the van der Pol equation in the form

$$\frac{d^2x}{dt^2} - \varepsilon\left[\frac{dx}{dt} - \frac{1}{3}\left(\frac{dx}{dt}\right)^3\right] + x = 0 \qquad (10.1)$$

and to have a small parameter in front of the second derivative.

Putting, in eq. (10.1)

$$\left.\begin{aligned} x &= \varepsilon\eta, \\ t &= \varepsilon t_1, \end{aligned}\right\} \qquad (10.2)$$

we obtain

$$\frac{1}{\varepsilon^2}\frac{d^2\eta}{dt_1^2} - \left[\frac{d\eta}{dt_1} - \frac{1}{3}\left(\frac{d\eta}{dt_1}\right)^3\right] + \eta = 0 \qquad (10.3)$$

or

$$\frac{1}{\varepsilon^2}\frac{d\dot{\eta}}{d\eta} = \frac{\dot{\eta} - \frac{1}{3}\dot{\eta}^3 - \eta}{\dot{\eta}}. \qquad (10.4)$$

For $\varepsilon \gg 1$, we may in first approximation neglect the summand $\frac{1}{\varepsilon^2}\frac{d\dot{\eta}}{d\eta}$ in eq. (10.4), after which we obtain the following relationship between $\dot{\eta}$ and η:

$$\dot{\eta} - \frac{1}{3}\dot{\eta}^3 = \eta, \qquad (10.5)$$

by which the character of the motion on the phase plane can be readily investigated.

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Let us construct the curve of eq.(10.5); (Fig.70). We note that, according to eq.(10.4), the field of directions on the curve of eq.(10.5) is horizontal, since $\frac{d\dot{\eta}}{d\eta} = 0$ for all values of $\dot{\eta}$ and η satisfying eq.(10.5). In the remaining points of the phase plane, however, except for points very close to the curve of eq.(10.5), the field of directions approaches the vertical as $\varepsilon \to \infty$ since, according to eq.(10.4), $\frac{d\dot{\eta}}{d\eta} \to \infty$ as $\varepsilon \to \infty$ for all points not satisfying eq.(10.5). Starting from this premise, it is obvious that, for large values of ε, the integral curve of eq.(10.4), leaving the arbitrary point P (cf. Fig.70), will be very close to a vertical straight line almost up to the point P_1, lying on the curve of eq.(10.5). Further, the integral curve runs along the curve of eq.(10.5), remaining below it until it reaches the neighborhood of the point P_2, after which it runs vertically upward until it again reaches the point of eq.(10.5). The integral curve will then follow along the curve of eq.(10.5), remaining above it; after reaching the point P_4, the integral curve will turn vertically downward. As a result, we obtain the limit cycle which, as $\varepsilon \to \infty$, will have the form shown in Fig.70.

We obtain such a picture because of the fact that the segments P_3P_4 and P_2P_5 of the curve of eq.(10.5) possess the property of attraction; in this case, the greater the value of ε, the stronger will be the attraction. Since the field of directions is vertical at any point while it is horizontal on the curve of eq.(10.5), each point will first asymptotically tend toward the curve of eq.(10.5) and will then move away from it, since the field of directions is horizontal along the curve; after this, the point will again tend to approach the curve of eq.(10.5). If ε is sufficiently great, these deviations will not be detectable so that, in practical cases, the picture shown in Fig.70 is obtained.

Let us find the asymptotic value for the period of oscillations in the approximation under consideration, by calculating the integral along the limit cycle.

For eq.(10.4) we have

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$$dt_1 = \frac{d\eta}{\dot{\eta}}, \tag{10.6}$$

whence

$$T_1 = \oint \frac{d\eta}{\dot{\eta}}. \tag{10.7}$$

Since, on the vertical parts of the cycle, $d\eta = 0$, we may replace eq.(10.7) by

$$T_1 = 2\int_{\dot{\eta}_1}^{\dot{\eta}_2} \frac{d\left(\dot{\eta} - \frac{1}{3}\dot{\eta}^3\right)}{\dot{\eta}} = 2\left(\ln \dot{\eta} - \frac{1}{2}\dot{\eta}^2\right)\Big|_{\dot{\eta}_1}^{\dot{\eta}_2}. \tag{10.8}$$

According to eq.(10.5) we find $\eta_1 = 1$, $\eta_2 = 2$ and, consequently, obtain the following equation for the period T_1 at large values of ε:

$$T_1 = 1{,}614 \tag{10.9}$$

or, using the old variables, the following asymptotic formula

$$T = 1{,}614\varepsilon. \tag{10.10}$$

Thus, for the case $\varepsilon \gg 1$ and using the asymptotic treatment, the oscillatory process will proceed as follows: When η increases, beginning with the values of $-\eta_0$, the velocity $\dot{\eta}$ will be positive and the representative point on the phase space will move along the curve P_3P_4 (cf. Fig.70). When η reaches its maximum value of $+\eta_0$, the representative point jumps from the position P_4 to P_5, which corresponds to an instantaneous reversal of sign of the velocity $\dot{\eta}$. When η then decreases, the velocity $\dot{\eta}$ will remain negative, and the representative point will move along the curve P_5P_2. At the point P_2, the velocity will again change its sign, while the representative point on the phase plane will jump to the position P_3.

Thus, during one period of oscillation, the velocity $\dot{\eta}$ will undergo a discontinuity twice, at the instance of reaching the maximum and minimum values of η. Of course, in reality the velocity is continuous (although fluctuating rapidly) since, even though ε is great, it is still a finite value, and, in speaking of a discontinuity, we introduce a certain simplification corresponding to the asymptotic approximation adopted by us.

After having obtained the relation between the velocity and the displacement on the phase plane and having found the period of oscillation, there will be no difficulty in constructing curves representing η and $\dot{\eta}$ as functions of t(Figs. 71-72).

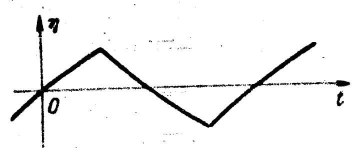

Fig. 71

The oscillations we have just considered are called relaxation oscillations and are widely found in nature.

The idealized discontinuous treatment of the van der Pol equations presented here for large values of ε may also be applied, in the general case, to the investigation of nonlinear oscillatory systems with $\varepsilon \gg 1$. In such a treatment, we neglect the inertia term in the equation, so that the relaxation oscillation will be characterized by the first-order differential equation

$$F\left(\frac{dx}{dt}\right) + x = 0, \tag{10.11}$$

It is convenient to invert this equation with respect to $\frac{dx}{dt}$ and to write it in the form

$$\frac{dx}{dt} = \Phi(x), \tag{10.12}$$

where $\Phi(x)$ represents a certain many-value function of the type schematically shown in Fig. 74.

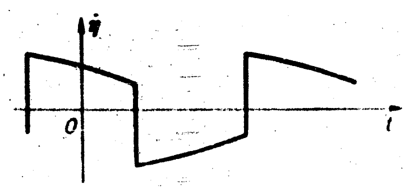

Fig. 72

We now present another example of a concrete relaxation oscillatory system described by an equation of the type of eq. (10.12).

Consider a circuit (Fig. 73) consisting of the inductance L, the resistance R, and a nonlinear element with an S-type volt-ampere characteristic, connected in

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series S to the source of direct-current voltage E_a. Here, for the S element, the volt-ampere characteristic has the form roughly shown in Fig.74. As a concrete model of such a nonlinear element, an electron tube in the dynatron state can be used.

On setting up a voltage balance for this type of circuit, we arrive at a differential equation of the type

$$L\frac{di}{dt}+Ri+v=E_a. \qquad (10.13)$$

Since the only source capable of supplying oscillation energy in our system is the inductance, the energy stored in it will be equal to $\frac{1}{2}Li^2$. Since the energy must

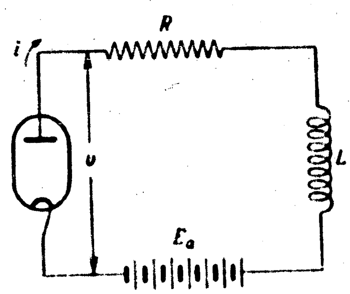

Fig.73

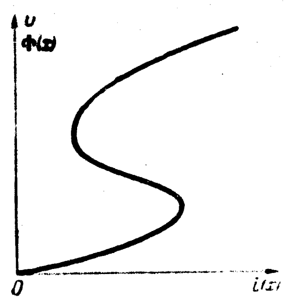

Fig.74

vary continuously during the course of the oscillatory process, it is evident that the magnitude of the current i must likewise vary continuously, smoothly increasing and decreasing.

On the other hand, Fig.74 shows clearly that, with a smooth increase in current i and, accordingly, with a smooth decrease, the voltage v will vary as shown in Fig.75. Let us denote the relation between the voltage and current, plotted in this diagram (taking into account only the segments of the solid line), by the functional relation

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$$v = f(i), \tag{10.14}$$

in which f (i) has two values for a current i varying in the interval (i_o, i_1). Let the parameters of the oscillator be so selected that in the interval (i_o, i_1) the values of the function

$$\Phi(i) = \frac{E_0 - Ri - f(i)}{L}, \tag{10.15}$$

corresponding to the lower branch of f (i) are positive, while those for the upper branch are negative. Then, in our diagram, a relaxation oscillatory process is excited, under whose effect the current i will fluctuate over the range from i_o to i_1. The differential equation describing the oscillatory process will be of the type of eq.(10.12).

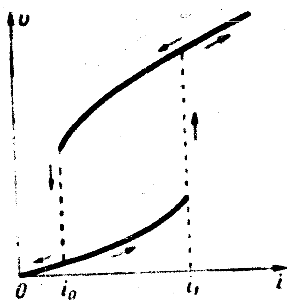

Fig.75

Obviously, in this example we did not carry the process as far as the construction of a second-order differential equation but immediately adopted a layout which led to a discontinuous equation of the type of eq.(10.12), in which the inertia term is disregarded.

Up to now we have considered the case of the presence of a single closed cycle. If the curve characterizing the relation (10.12) has the form shown in Fig.76, then we obtain two closed cycles.

We note that the relaxation oscillatory processes under consideration take place without external periodic forces; for this reason, it is natural to call eq.(10.12) the equation of free relaxation oscillations.

The above-described discontinuous treatment of relaxation oscillatory processes is convincing per se.

If, however, a rigorous justification is desired or if the appropriate cor-

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rections for the form of oscillation and for their period T are to be calculated, the rigorous method of asymptotic approximation developed by A.A.Dorodnitsyn must be used.

Section 11. A.A.Dorodnitsyn's Method for the van der Pol Equation

In this Section we present the method developed by A.A.Dorodnitsyn (Bibl.14), with which the integral curves may be constructed on the phase plane if $\varepsilon \gg 1$.

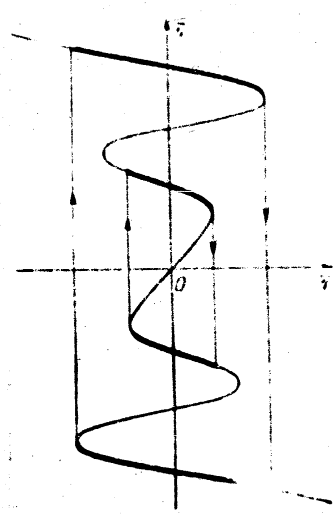

Fig.76

The essence of this method consists in the introduction of certain "connecting" regions and in the construction, for these overlapping regions, of special asymptotic expansions in powers of $\frac{1}{\varepsilon}$.

To make the application of this method absolutely clear, we will use A.A.Dorodnitsyn's method for developing a solution of the van der pol equation

$$\frac{d^2x}{dt^2} - \varepsilon(1 - x^2)\frac{dx}{dt} + x = 0 \qquad (11.1)$$

for large values of the parameter ε*.

On the phase plane (x, y), where $y = \frac{dx}{dt}$, this equation is transformed into the following form:

$$y\frac{dy}{dx} - \varepsilon(1 - x^2)y + x = 0. \qquad (11.2)$$

As is commonly known, the limit cycle for eq.(11.2) has the form shown in the diagram of Fig.77.

At large values of the parameter ε, the solution of eq.(11.2) in the regions I and III tends respectively toward the solutions of the "abbreviated" equations

* We note that V.V.Kazakevich (Bibl.50) has developed an interesting method of investigating eq.(11.1).

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$$y\frac{dy}{dx} - \varepsilon(1-x^2)y = 0, \tag{11.3}$$

$$-\varepsilon(1-x^2)y + x = 0. \tag{11.4}$$

The regions I and III, however, in which eq.(11.2) can be replaced by eqs.(11.3) and (11.4), do not osculate so that the solutions of these equations cannot be conjugated since we do not know how to select the integration constant in eq.(11.3). Consequently, at an analytic continuation of the solution into region III, this solution changes into a solution tending toward the solution of eq.(11.4).

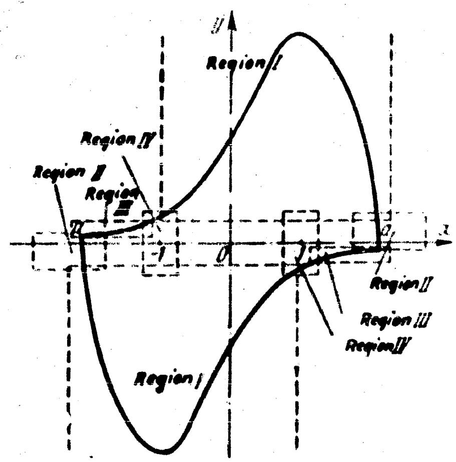

Fig. 77

For such a conjugation of the solutions to be practicable, the two "osculating" regions II and IV are introduced, for which the asymptotic solutions are constructed directly for eq.(11.2), since in these regions we cannot make use of the "abbreviated" equations. The regions I, II, III, and IV overlap, so that a solution of each cycle can be obtained with an accuracy to terms of any desired order of smallness of $\frac{1}{\varepsilon}$.

Let us now construct the asymptotic solutions of eq.(11.2) for the regions so introduced (cf. Fig.77). In this case, in view of the existing symmetry, we may limit our consideration to only one part of each of the introduced regions.

Let us first construct the solution for region I. For this purpose, let us denote the values of x for which $\frac{dy}{dt} = 0$ (for the limit cycle $a_1 = a_2 = a$) by a_1 and a_2. Then, region I is determined by the following inequalities:

$$\left.\begin{array}{l} -1+\nu < x < a_1-\nu, \quad y>0, \quad \nu>0; \\ -a_2+\nu < x < 1-\nu, \quad y<0, \quad \nu>0. \end{array}\right\} \tag{11.5}$$

As pointed out above, it is sufficient to find the solution only for the first part of the region of eq.(11.5). Let us seek this solution in the form of the series

$$y = \varepsilon \sum_{n=0}^{\infty} \varepsilon^{-2n} f_n(x). \tag{11.6}$$

On substituting the value y from eq.(11.6) in eq.(11.3) and equating the coefficients of the same powers of ε, we obtain a system of equations from which we find successively all $f_n(x)$ ($n = 0, 1, 2, \ldots$).

Thus, for the first two functions, we have

$$f_0(x) = c + x - \frac{1}{3}x^3, \tag{11.7}$$

$$f_1(x) = \frac{x_1}{x_1^2-1}\left[\ln\left(1-\frac{x}{x_1}\right) - \frac{1}{2}\ln\frac{(2x+x_1)^2+3(x_1^2-4)}{4(x_1^2-3)}\right] + \frac{x_1^2-2}{x_1^2-1}\sqrt{\frac{3}{x_1^2-4}}\left[\operatorname{arctg}\frac{2x+x_1}{\sqrt{3(x_1^2-4)}} - \operatorname{arctg}\frac{x_1}{\sqrt{3(x_1^2-4)}}\right], \tag{11.8}$$

where x_1 denotes a real positive root of the equation $f_0(x) = 0$, assuming that $c > \frac{2}{3}$, which is in fact the case for the limit cycle. The functions $f_n(x)$ have singularities in the neighborhood of the points $x = x_1$, but the series of eq.(11.6) preserves its asymptotic character up to values of x satisfying the condition

$$O(x_1 - x) > O\left(\frac{\ln \varepsilon}{\varepsilon^2}\right).$$

In particular, the series (11.6) is an asymptotic series when $x = x_1 - O\,\frac{1}{\varepsilon}$;

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in this case, y will be of the order of unity.

Let us now find the asymptotic solution for the region II, which is the neighborhood of the points $y = 0$, $x = a_1$; $y = 0$, $x = -a_2$. To be more definite, we will consider the portion of the region II for which $y = 0$, $x = a_1$.

Let us introduce the new variable z from the formula $z = -\varepsilon y$ and find x as a function of z. Equation (11.2) is written as follows:

$$\frac{dx}{dz} = \frac{1}{\varepsilon^2}\frac{z}{z(x^2-1)-x}. \tag{11.9}$$

Let us seek the solution of this equation in the form of the series

$$x = \sum_{n=0}^{\infty} \chi_n(z)\,\varepsilon^{-2n}. \tag{11.10}$$

On substituting the value of x from eq.(11.10) in eq.(11.9) and equating the coefficients of equal powers of ε, we obtain a system of equations for determining the functions $\chi_n(z)$ $(n = 0, 1, 2, \ldots)$. We then have the following expressions for the first two functions:

$$\chi_1(z) = \frac{1}{a_1^2-1}\left[z + \frac{a_1}{a_1^2-1}\ln\left(1 - \frac{a_1^2-1}{a_1}z\right)\right], \tag{11.11}$$

$$\chi_2(z) = \frac{a_1}{(a_1^2-1)^4}\left\{(a_1^2-1)z\left(z + \frac{a_1^2+1}{a_1(a_1^2-1)}\right) + \left[\frac{a_1^2+1}{a_1^2-1} + 2a_1 z - 2(a_1^2-1)z^2\right]\frac{\ln\left[1 - \frac{z(a_1^2-1)}{a_1}\right]}{1 - z\frac{a_1^2-1}{a_1}} + \frac{3a_1^2+1}{2(a_1^2-1)}\ln^2\left(1 - z\frac{a_1^2-1}{a_1}\right)\right\}. \tag{11.12}$$

These functions have singularities as $z \to \frac{a_1}{a_1^2-1}$ and as $z \to -\infty$. The series of eq.(11.10), however, maintains its asymptotic character for all values of z satisfying the condition

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$$O\left(\frac{a_1}{a_1^2-1} - z\right) > O\left(\frac{\ln \varepsilon}{\varepsilon^2}\right),$$

and, in the case $z < 0$, also for all values of z satisfying the condition

$$O(z) < O(\varepsilon^2).$$

In this case, the series (11.10) will be asymptotically convergent for $z = -\varepsilon$, i.e., for $y = 1$.

Since the series of eqs.(11.6) and (11.10) converge asymptotically for the same values of x at which $y = O(1)$, they may be conjugated. For this purpose, we must determine the constant a_1 for an assigned value of the constant c. Putting $y = 1$ in eqs.(11.6) and (11.10), we obtain the following two equations in the two unknowns x^* and a_1:

$$1 = \varepsilon \sum_{n=0}^{\infty} f_n(x^*)\varepsilon^{-2n}, \quad x^* = \sum_{n=0}^{\infty} \chi_n(-\varepsilon)\varepsilon^{-2n}. \qquad (11.13)$$

We find x^* from the first equation and a_1 from the second equation, expressed in terms of c or in terms of x_1.

Next, the solution for region III will be defined. This region is determined by the following intervals:

$$\left.\begin{array}{l} a_1 - \nu > x > 1 + \nu, \quad y < 0, \quad \nu > 0; \\ -a_2 + \nu < x < -1 - \nu, \quad y > 0, \quad \nu > 0 \end{array}\right\} \qquad (11.14)$$

and is of substantial importance for relaxation oscillations. When x enters the region of eq.(11.14), the oscillatory system immediately changes, with a high degree of accuracy, into steady self-sustained oscillations.

We will now consider that part of the region III for which $y < 0$.

By performing several calculations, it can be shown that the solution of eq.(11.2) in this region is obtainable in the form of the series

$$y = -\frac{1}{\varepsilon}\sum_{n=0}^{\infty} P_n(x)\varepsilon^{-2n}, \qquad (11.15)$$

where $P_n(x)$ is determined as above from a series of recurrent equations and has the form

$$P_0(x) = \frac{x}{x^2 - 1}, \qquad P_1(x) = -\frac{x(x^2+1)}{(x^2-1)^4}, \ldots \tag{11.16}$$

The series (11.15) maintains its asymptotic character under the condition $O(x - 1) > O(\varepsilon^{-2/3})$ while, on approach to the boundary of convergence, the value y in eq.(11.15) will be of the order of $\varepsilon^{-1/3}$.

Next, the solution in region IV will be derived. This region is defined as follows:

$$\left.\begin{array}{ll} 1 - \nu < x < 1 + \nu, & p < 0, \quad \nu > 0; \\ -1 - \nu < x < -1 + \nu, & p > 0, \quad \nu > 0. \end{array}\right\} \tag{11.17}$$

As shown above, on approach to the boundary of region III, y approaches $O(\varepsilon^{-1/3})$. It is, therefore, natural to introduce the following substitution of variables:

$$y = -\varepsilon^{-1/3} Q(u), \quad u = \varepsilon^{2/3}(x - 1). \tag{11.18}$$

Then eq.(11.2) will assume the following form:

$$Q\frac{dQ}{du} - 2uQ + 1 = \varepsilon^{-2/3}(u^2Q - u). \tag{11.19}$$

The solution of this equation will be derived again in the form of the asymptotic series

$$Q(u) = \sum_{n=0}^{\infty} Q_n(u)\,\varepsilon^{-2/3 n}, \tag{11.20}$$

which we then substitute in eq.(11.19); by then equating the coefficients of equal powers of ε, we obtain a series of equations for the successive determination of the functions $Q_n(u)$ ($n = 0, 1, 2, \ldots$). In this case, the initial condition for $Q_n(u)$ must be determined in such a way that the obtained solution is conjugate with the solution in the region III.

After several calculations, we find

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$$Q_0(u) = u^2 + \alpha + \frac{1}{u} - \frac{\alpha}{3} - \frac{1}{u^3} - \frac{1}{4u^4} + \frac{\alpha}{5u^5} + \dots$$

$$Q_1(u) = \frac{1}{A(u)}\left[C + \int_0^u A(u)\left(u^2 - \frac{u}{Q_0}\right)du\right],$$

$$A(u) = \exp\left(-\int_0^u \frac{du}{Q_0^2}\right), \qquad (11.21)$$

where α is the smallest root of the equation

$$J_{1/3}\left(\frac{2}{3}\alpha^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}\alpha^{3/2}\right) = 0.$$

For conjugation with the solution y of eq.(11.15), determined for the region III, it is necessary that the quantity $\varepsilon^{-2/3} Q_1(u)$ is bounded for $u = Q(\varepsilon^{\gamma})$.

An analysis of the expression for $Q_n(u)$ shows that the series (11.20) maintains its asymptotic character up to values of u limited by the condition $Q(u) < Q(\varepsilon^{2/3})$, i.e., at values of x satisfying the condition $O(x - 1) < O(1)$, and, thus, of the region in which the suitable solutions of eq.(11.20) and (11.15) overlap.

It now remains for us to conjugate the solutions for the regions I and IV. For this purpose we must conjugate the solution of eq.(11.6) with the solution of eq.(11.20), taking into account, in the latter, the substitution of variables(11.18).

We note that since $y > 0$ for $x = -1$, the constant c must be more than $\frac{2}{3}$.

Let us put $c = \frac{2}{3} + \gamma$, and let us determine the order of γ. Since $y(-1) = O(\varepsilon^{-1/3})$, it follows that $\varepsilon\gamma$ will likewise be of the order of $\varepsilon^{-1/3}$ and, consequently, $\gamma = O(\varepsilon^{-4/3})$.

It is easy to show that the series (11.6) maintains its asymptotic character up to values of x satisfying the condition $O(x + 1) > O(\varepsilon^{-1/3})$; in this way, the regions in which the solutions (11.20) and (11.6) are valid overlap, with an asymptotic convergence of these expansions being ensured for $x = -1 + \varepsilon^{-1/3}$.

Thus, the integration constant c can be determined by equating, for $x = -1 +$

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$+ \varepsilon^{-1/3}$, the values of y obtained from eqs.(11.20) and (11.6):

$$\varepsilon^{-1/3} \sum_{n=0}^{\infty} \varepsilon^{-2/3 n} Q_n(-\varepsilon^{-1/3}) = \varepsilon \sum_{n=0}^{\infty} \varepsilon^{-2n} f_n(-1 + \varepsilon^{-1/3}). \quad (11.22)$$

On determining γ from this relation with an accuracy to terms of the order of $\varepsilon^{-2/3}$, we obtain

$$\gamma \approx \alpha\varepsilon^{-4/3} - \frac{4}{9}\frac{\ln \varepsilon}{\varepsilon} + \left(b_0 - 1 - \frac{2}{3}\ln\frac{3}{2}\right)\varepsilon^{-2} + O(\varepsilon^{-8/3}), \quad (11.23)$$

where $b_0 = b_0(\alpha)$ is a known quantity.

On determining the constant $c = \frac{2}{3} + \gamma$, it is easy to find x_1 (the root of the equation $f_0(x_1) = 0$) after which, making use of eq.(11.12), we find the amplitude of the self-sustained oscillations

$$a \approx 2 + \frac{\alpha}{3}\varepsilon^{-4/3} - \frac{16}{27}\frac{\ln \varepsilon}{\varepsilon^2} + \frac{1}{9}(3b_0 - 1 + 2\ln 2 - 8\ln 3)\varepsilon^{-2} + O(\varepsilon^{-8/3}). \quad (11.24)$$

The period of the self-sustained oscillations may be calculated according to the formula

$$T = 2\int_{-a}^{a} \frac{dx}{y(x)}. \quad (11.25)$$

For this purpose, we divide the entire interval of integration into five parts corresponding to the different regions:

1) from $-a$ to $-x_2$, in the region II, where x_2 is the value of x obtained from eq.(11.10), when

$$z = \frac{(1 - \varepsilon^{-4/3})a}{a^2 - 1};$$

2) from $-x_2$ to $-(1 + \varepsilon^{-1/3})$, in the region III;

3) from $-(1 + \varepsilon^{-1/3})$ to $-(1 - \varepsilon^{-1/3})$ in the region IV;

4) from $-(1 - \varepsilon^{-1/3})$ to x^* in the region I, where x^* is determined by the

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formula

$$x^{*}=x_1-\frac{1}{\varepsilon}\frac{1}{x_1^2-1}-\frac{\ln\varepsilon}{\varepsilon^2}\frac{x_1}{(x_1^2-1)^2}-$$
$$-\frac{1}{\varepsilon^2}\left[\frac{x_1}{(x_1^2-1)^2}\ln x_1(x_1^2-1)-\frac{g}{x_1^2-1}+\frac{x_1}{(x_1^2-1)^3}\right]-$$
$$-\frac{\ln\varepsilon}{\varepsilon^3}\frac{2x_1}{(x_1^2-1)^4}+O(\varepsilon^{-3}); \qquad (11.26)$$

5) from x^{*} to a in the region II.

Then the total period T will be equal to

$$T=2[T_1+T_2+T_3+T_4+T_5], \qquad (11.27)$$

where T_i is a part of the integral of eq.(11.25) taken with the i^{th} interval of integration.

By performing the integration for the total period, we obtain

$$T\approx(3-2\ln 2)\varepsilon+3\gamma\varepsilon^{-1/3}-\frac{22}{9}\frac{\ln\varepsilon}{\varepsilon}+$$
$$+\left(3\ln 2-\ln 3-\frac{1}{6}+b_0-2d\right)\varepsilon^{-1}+O(\varepsilon^{-4/3}) \qquad (11.28)$$

or, on substituting the numerical values of the coefficients,

$$T\approx 1{,}613706\varepsilon+7{,}01432\varepsilon^{-1/3}-\frac{22}{9}\frac{\ln\varepsilon}{\varepsilon}+$$
$$+0{,}0087\,\varepsilon^{-1}+O(\varepsilon^{-4/3}). \qquad (11.29)$$

For a sufficiently large value of ε, all terms except the first in this equation may be neglected, resulting in an asymptotic expression for a period coinciding with eq.(10.10) given in the preceding Section.

For the asymptotic solutions (11.6), (11.10), (11.15), and (11.20), it may be proved without difficulty, by the method of successive approximation, that they are convergent (asymptotically) in their respective regions.

A consideration of the above asymptotic equations clearly indicates that the case of a large ε is considerably more complex than the case of a small ε. For $\varepsilon \ll 1$, we had net power asymptotic formulas while, for $\varepsilon \gg 1$, fractional powers or

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logarithmic terms enter the equation. In the case of $\varepsilon \gg 1$, we have a higher sensitivity to the specific form of the equation than in the case of $\varepsilon \ll 1$. It is therefore natural that, at high nonlinearity, the actual construction of approximate solutions should require a higher degree of concreteness in the differential equations under study.

We note that for investigating this important and difficult problem of finding the asymptotic approximation for a large parameter (or for a small parameter before a higher derivative) the effective asymptotic methods developed by A.N.Tikhonov (Bibl.40) and his students may be used successfully.

CHAPTER III

THE INFLUENCE OF EXTERNAL PERIODIC FORCES

Section 12. Asymptotic Expansions in the "Nonresonant" Case

We will now discuss oscillatory systems under the influence of external periodic forces, depending explicitly on the time.

We will consider a system with one degree of freedom for which the differential equation of motion may be represented in the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(\nu t,\ x,\ \frac{dx}{dt}\right), \qquad (12.1)$$

where ε is a small positive parameter and $f(\nu t,\ x,\ \frac{dx}{dt})$ is a function which is periodic with respect to νt and has the period 2π; this can be represented in the form

$$f\left(\nu t,\ x,\ \frac{dx}{dt}\right) = \sum_{n=-N}^{N} e^{in\nu t} f_n\left(x,\ \frac{dx}{dt}\right). \qquad (12.2)$$

We assume that the coefficients $f_n(x,\ \frac{dx}{dt})$ in the finite sum of eq.(12.2) are certain polynomials with respect to x and $\frac{dx}{dt}$.

Equation (12.1) under consideration may obviously be interpreted as the equation of oscillation of a certain mechanical system of unit mass with the natural frequency ω under the influence of the small nonlinear disturbance $\varepsilon f(\nu t,\ x,\ \frac{dx}{dt})$, explicitly depending on the time. We have already become familiar, from the Introduction, with numerous examples of oscillatory systems described by an equation of this form.

Before passing to a discussion of the method of finding asymptotic solutions for a system described by eq.(12.1), let us discuss again the analysis of the in-

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fluence of periodic actions on the system, starting from physical considerations.

In the absence of any perturbation, i.e., at $\varepsilon = 0$, we obtain the purely harmonic oscillations

$$x = a\cos(\omega t + \varphi),$$
$$\frac{dx}{dt} = -a\omega\sin(\omega t + \varphi),$$

where a and φ are arbitrary constants.

Obviously, if the method set forth in the preceding Chapter is used for determining the functions u_1, $u_2, \ldots$, then, since the external influence depends on the time, terms containing $\sin(n\nu + m\omega)t$ and $\cos(n\nu + m\omega)t$, where n and m are integers, must appear in the expansion of the function $\varepsilon f(\nu t, x, \frac{dx}{dt})$ into a Fourier series (after substitution in it of $x = a\cos(\omega t + \varphi)$, $\frac{dx}{dt} = -a\omega\sin(\omega t + \varphi)$), owing to the periodicity of this series in νt. Thus, harmonic components with compound frequencies of the form $(n\nu + m\omega)$ appear on the right sides of the differential equations determining u_1, $u_2, \ldots$.

It is entirely clear that, when one of such compound frequencies is close to the natural frequency of the system, the corresponding harmonic of the disturbing force will be able to exercise a considerable influence on the character of the oscillation, even if the corresponding coefficient in the expression of the applied perturbation is small (the amplitude of the corresponding harmonic being small). Of course, the smaller this coefficient, the smaller must be the detuning between the natural and external frequency before this influence makes itself felt. Thus, as established above, resonant phenomena occur in nonlinear oscillatory systems not only at $\omega \approx \nu$, as in the ordinary linear systems, but also in the case where one of the compound frequencies of the external influence is close to the natural frequency of the system, i.e., if $n\nu + m\omega \approx \nu$.

Thus, in nonlinear systems, resonance may occur when the condition

$$\nu \approx \frac{p}{q}\omega, \tag{12.3}$$

is satisfied, where p and q are relatively prime integers (usually small).

Let us introduce the following classification for the various cases of reso-

nance:

1) $p = q = 1$, i.e., $\nu \approx \omega$; we shall term this case the "main" or ordinary resonance;

2) $q = 1$, i.e., $\nu \approx p\omega$ or $\omega \approx \frac{\nu}{p}$; we will call this case resonance on the overtone of natural frequency or submultiple resonance (fractional, since the oscillation here proceeds at a frequency equal to a fraction of the external frequency) or parametric resonance. Resonance of this type is possible also in linear systems with periodic coefficients;

3) $p = 1$, i.e., $\omega \approx q\nu$; we will call this case resonance on the overtone of the external frequency.

Here the following circumstance must be noted: Since p and q can assume the values of all possible integers, the set $\{\frac{p}{q}\}$ is dense, and, consequently, the ratio $\frac{p}{q}$, by means of an appropriate choice of the numbers p and q, may be brought close to any predetermined number. For this reason, we might get the impression that resonance in a nonlinear system is possible at arbitrary values of p and q. In reality, this is not the case, since not all the possibilities indicated by eq.(12.3) can actually be realized or, in other words, it is not at all values of p and q that the corresponding resonance does occur. In practice, the expansion (12.2) has a finite number of terms, and the numbers p and q are entirely determined by the character of the oscillatory system under study.

Next, let us ascertain what resonances appear in the first approximation.

As usual, we will assume that the oscillations in first approximation remain purely harmonic in form and that, at each individual cycle, they may with sufficient accuracy be approximated by an ordinary harmonic; however, a small disturbing force, no matter how complex its structure, may affect the course of the oscillations by producing only slow but nevertheless systematic variations in the amplitude and phase of the oscillation (slow in comparison with the natural unit of time, the period of a single cycle).

From the definition of resonance, we may consider that resonance is characterized precisely by the fact that a small disturbing force may lead to a considerable,

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and often very great, change in the oscillation amplitude. This occurs only when the work performed by the external force throughout a cycle of oscillations is not destroyed since, otherwise, the external force would cause only small vibrations.

The expression of the disturbing force $\epsilon f(\nu t,\ x,\ \frac{dx}{dt})$ in a state of harmonic oscillation [i.e., for $x = a\cos(\omega t + \varphi)$, $\frac{dx}{dt} = -a\omega\sin(\omega t + \varphi)$] contains, as stated above, various harmonics with frequencies of $\pm n\nu \pm m\omega$.

Let us set up an expression for the virtual work that would be performed by this disturbing force in a state of harmonic oscillations, over the virtual displacements

$$\delta x_0 = \delta a\cos(\omega t + \varphi) - \delta\varphi a\sin(\omega t + \varphi), \qquad (12.4)$$

corresponding to the virtual increment of amplitude and phase of oscillation.

For this calculation, it is convenient to represent the expression for the virtual work in a state of harmonic oscillations

$$\epsilon f\left(\nu t,\ x_0,\ \frac{dx_0}{dt}\right)\delta x_0 \qquad (12.5)$$

by the aid of a Fourier series in the form of a sum of harmonic terms of the following frequencies

$$\lambda_{nm} = n\nu + m\omega.$$

However, on forming the mean of this sum over a sufficiently long time interval, only those terms in which the frequencies λ_{nm} are sufficiently small will remain perceptible.

Thus, in first approximation, only those resonances will appear for which the frequencies in the expression of virtual work (12.5) are sufficiently close to zero. Of course, the intensity of the resonance will be weaker, the smaller the corresponding amplitude in eq.(12.5).

After these preliminary remarks, let us pass to the formulation of the methods of actually constructing the approximate solutions.

Let us begin by considering the oscillatory system described by eq.(12.1) for the nonresonant case, as the simplest one. That is, we will assume that there are no compound frequencies ($n\nu + m\omega$), entering into this approximation that are equal

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(or close to) the frequency ω:

$$n\nu + m\omega \approx \omega. \qquad (12.6)$$

Here we must point out a fact well known in the theory of numbers. If $\frac{\nu}{\omega}$ is irrational, it is always possible to select integers n and m such that the expression

$$n\nu + (m-1)\omega$$

will be as close to zero as desired.

For this reason, if the expression for the approximate solution under consideration contains harmonics with all linear combinations $n\nu + m\omega$, then we must impose the condition that the ratio $\frac{\nu}{\omega}$ does not approximate too rapidly a rational number and will not make the considered expression divergents (cf. above on this subject).

In our approach to the construction of an approximate solution of the differential equation (12.1) we will use the very same intuitive considerations as in the case of a disturbance which does not contain the time explicitly.

In the complete absence of disturbing forces ($\varepsilon = 0$), the oscillations will obviously be purely harmonic $x = a \cos \psi$, with constant amplitude and uniformly rotating phase angle $\frac{da}{dt} = 0$, $\frac{d\psi}{dt} = \omega$.

The influence of the perturbation is expressed as follows: First, it may appear in the oscillations as an overtone or as a harmonic of compound frequencies of various orders of smallness; therefore the solution must be sought in the form

$$x = a \cos \psi + \varepsilon u_1(a, \psi, \nu t) + \varepsilon^2 u_2(a, \psi, \nu t) + \dots, \quad (12.7) \qquad (12.7)$$

where the functions $u_1(a, \psi, \nu t)$, $u_2(a, \psi, \nu t), \dots,$ are periodic with respect to both angular variables, with the period 2π.

Secondly, the amplitude and the rotary phase velocity can now no longer be constant, but must be determined, as in the last Chapter, by the differential equations

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \dots, \\ \frac{d\psi}{dt} &= \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \dots \end{aligned}\right\} \qquad (12.8)$$

The right sides of these equations must depend only on the amplitude, since, in the absence of resonance, the phase of the natural oscillations is not related to the phase of the external forces, and therefore the latter phase has no effect on

the amplitude of the oscillations or on the full phase of the oscillation. Of course, in the resonance case we will have to introduce the dependence on the phase shift in the expressions for both the instantaneous frequency and the instantaneous amplitude.

Thus, the problem of constructing approximate solutions for eq.(12.1) in the nonresonant case reduces down to a problem analogous to that considered in the first Section: It is required to find functions

$$u_1(a, \psi, \nu t), \quad u_2(a, \psi, \nu t), \ \ldots, \quad A_1(a), \quad A_2(a), \ \ldots,$$
$$B_1(a), \quad B_2(a), \ \ldots$$

such that eq.(12.7) in which the functions of time defined by eq.(12.8) have been substituted for a and ψ, will be a solution of our original eq.(12.1).

As in Section 1, after solving this problem, i.e. after finding explicit expressions for the coefficients of the expansion in the right-hand sides of eqs.(12.7) and (12.8), the question of the integration of eq.(12.1) reduces to the simpler question of integrating eq.(12.8). It must be noted that, in the nonresonant case, we obtain equations with separable variables for the determination a and ψ; in the resonant cases, as demonstrated below, the variables in these equations will no longer be separated in the general case.

Before proceeding to construct the functions $u_1(a, \psi, \nu t)$, $u_2(a, \psi, \nu t), \ldots,$ $A_1(a)$, $A_2(a), \ldots,$ $B_1(a)$, $B_2(a), \ldots,$ certain additional conditions must, as above, be imposed, so that the coefficients of the expansions (12.8) will be uniquely determined.

As these conditions it is natural to take the conditions of the absence of resonant terms in the functions $u_1(a, \psi, \nu t)$, $u_2(a, \psi, \nu t), \ldots,$ i.e., of terms whose denominators may vanish.

This condition is equivalent to the requirement that the first harmonic of the argument ψ must be absent from the functions $u_1(a, \psi, \nu t)$, $u_2(a, \psi, \nu t), \ldots,$ and, from the physical point of view, must correspond to the choice of the full amplitude of the fundamental harmonic of the oscillation as the quantity a.

After these preliminary remarks, let us proceed to determining the functions

$u_1(a, \psi, \nu t)$, $u_2(a, \psi, \nu t), \ldots$, $A_1(a)$, $A_2(a), \ldots$, $B_1(a)$, $B_2(a), \ldots$, taking the above-stated additional condition into consideration.

Differentiating eq.(12.7), we have

$$\frac{dx}{dt} = \left\{\cos\psi + \varepsilon\frac{\partial u_1}{\partial a} + \varepsilon^2\frac{\partial u_2}{\partial a} + \cdots\right\}\frac{da}{dt} + \left\{-a\sin\psi + \right.$$
$$\left. + \varepsilon\frac{\partial u_1}{\partial \psi} + \varepsilon^2\frac{\partial u_2}{\partial \psi} + \cdots\right\}\frac{d\psi}{dt} + \varepsilon\frac{\partial u_1}{\partial t} + \varepsilon^2\frac{\partial u_2}{\partial t} + \cdots, \qquad (12.9)$$

$$\frac{d^2x}{dt^2} = \left\{\cos\psi + \varepsilon\frac{\partial u_1}{\partial a} + \varepsilon^2\frac{\partial u_2}{\partial a} + \cdots\right\}\frac{d^2a}{dt^2} +$$
$$+ \left\{-a\sin\psi + \varepsilon\frac{\partial u_1}{\partial \psi} + \varepsilon^2\frac{\partial u_2}{\partial \psi} + \cdots\right\}\frac{d^2\psi}{dt^2} + \left\{\varepsilon\frac{\partial^2 u_1}{\partial a^2} + \right.$$
$$\left. + \varepsilon^2\frac{\partial^2 u_2}{\partial a^2} + \cdots\right\}\left(\frac{da}{dt}\right)^2 + \left\{-a\cos\psi + \varepsilon\frac{\partial^2 u_1}{\partial \psi^2} + \varepsilon^2\frac{\partial^2 u_2}{\partial \psi^2} + \cdots\right\} \times$$
$$\times \left(\frac{d\psi}{dt}\right)^2 + 2\left\{\varepsilon\frac{\partial^2 u_1}{\partial a\,\partial t} + \varepsilon^2\frac{\partial^2 u_2}{\partial a\,\partial t} + \cdots\right\}\frac{da}{dt} + 2\left\{\varepsilon\frac{\partial^2 u_1}{\partial t\,\partial \psi} + \right. \qquad (12.10)$$
$$\left. + \varepsilon^2\frac{\partial^2 u_2}{\partial t\,\partial \psi} + \cdots\right\}\frac{d\psi}{dt} + 2\left\{-\sin\psi + \varepsilon\frac{\partial^2 u_1}{\partial a\,\partial \psi} + \varepsilon^2\frac{\partial^2 u_2}{\partial a\,\partial \psi} + \cdots\right\} \times$$
$$\times \frac{da}{dt}\frac{d\psi}{dt} + \varepsilon\frac{\partial^2 u_1}{\partial t^2} + \varepsilon^2\frac{\partial^2 u_2}{\partial t^2} + \cdots$$

After substituting $\frac{da}{dt}$, $\frac{d^2a}{dt^2}$, $\frac{d\psi}{dt}$, $\frac{d^2\psi}{dt^2}$ in eq.(12.10) by their values in eqs.(12.8) and (1.10) of Section 1, we substitute the resultant values for $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$ and also eq.(12.7) in the left side of eq.(12.1); then, after arranging the result by powers of the small parameter ε, we get

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon\left\{\frac{\partial^2 u_1}{\partial \psi^2}\omega^2 + \frac{\partial^2 u_1}{\partial t^2} + 2\frac{\partial^2 u_1}{\partial \psi\,\partial t}\omega + \omega^2 u_1 - \right.$$
$$\left. - 2a\omega B_1\cos\psi - 2\omega A_1\sin\psi\right\} + \varepsilon^2\left\{\frac{\partial^2 u_2}{\partial \psi^2}\omega^2 + \frac{\partial^2 u_2}{\partial t^2} + \right.$$
$$+ 2\frac{\partial^2 u_2}{\partial \psi\,\partial t}\omega + \omega^2 u_2 - 2a\omega B_2\cos\psi - 2\omega A_2\sin\psi + \qquad (12.11)$$
$$+ \left(A_1\frac{dA_1}{da} - aB_1^2\right)\cos\psi - \left(aA_1\frac{dB_1}{da} + 2A_1B_1\right)\sin\psi +$$

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$$+2\omega A_1\frac{\partial^2 u_1}{\partial a\,\partial\psi}+2\omega B_1\frac{\partial^2 u_1}{\partial\psi^2}+2\frac{\partial^2 u_1}{\partial a\,\partial t}A_1+2\frac{\partial^2 u_1}{\partial\psi\,\partial t}B_1\Big\}+\varepsilon^3\ldots$$

The right side of eq.(12.1), according to eqs.(12.7) and (12.9), can be represented in the form

$$\left.\begin{aligned}\varepsilon f\left(\nu t,\ x,\ \frac{dx}{dt}\right)&=\varepsilon f(\nu t,\ a\cos\psi,\ -a\omega\sin\psi)+\\&+\varepsilon^2\left[f'_x(\nu t,\ a\cos\psi,\ -a\omega\sin\psi)\,u_1+\right.\\&+f'_{x'}(\nu t,\ a\cos\psi,\ -a\omega\sin\psi)\times\\&\times\left.\left(A_1\cos\psi-aB_1\sin\psi+\frac{\partial u_1}{\partial\psi}\omega+\frac{\partial u_1}{\partial t}\right)\right]+\varepsilon^3\ldots\end{aligned}\right\}\qquad(12.12)$$

In order for the required expression (12.7) to satisfy the initial eq.(12.1) with an accuracy to quantities of the order of smallness of ε^{m+1} (as above, we confine ourselves to finding the m^{th} approximation), the coefficients of the same powers of ε in the right-hand sides of eqs.(12.11) and (12.12), to terms of the m^{th} order inclusive, must be equated.

As a result we obtain a system of m equations for the determination of $u_1(a,\ \psi,\ \nu t)$, $u_2(a,\ \psi,\ \nu t),\ldots,\ u_m(a,\ \psi,\ \nu t)$, and also of $A_1(a),\ A_2(a),\ldots,\ A_m(a),\ B_1(a),\ B_2(a),\ldots,\ B_m(a)$:

$$\begin{aligned}\omega^2\frac{\partial^2 u_1}{\partial\psi^2}+2\omega\frac{\partial^2 u_1}{\partial\psi\,\partial t}+\frac{\partial^2 u_1}{\partial t^2}+\omega^2 u_1&=\\=f_0(a,\ \psi,\ \nu t)+2a\omega B_1\cos\psi+2\omega A_1\sin\psi,&\end{aligned}\qquad(12.13)$$

$$\left.\begin{aligned}\omega^2\frac{\partial^2 u_2}{\partial\psi^2}+2\omega\frac{\partial^2 u_2}{\partial\psi\,\partial t}+\frac{\partial^2 u_2}{\partial t^2}+\omega^2 u_2&=\\=f_1(a,\ \psi,\ \nu t)+2a\omega B_2\cos\psi+2\omega A_2\sin\psi,&\\\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\\omega^2\frac{\partial^2 u_m}{\partial\psi^2}+2\omega\frac{\partial^2 u_m}{\partial\psi\,\partial t}+\frac{\partial^2 u_m}{\partial t^2}+\omega^2 u_m&=\\=f_{m-1}(a,\ \psi,\ \nu t)+2a\omega B_m\cos\psi+2\omega A_m\sin\psi,&\end{aligned}\right\}\qquad(12.14)$$

where, for abbreviation, we introduce the symbols

$$
\begin{aligned}
&f_0(a,\ \psi,\ \nu t) = f(\nu t,\ a\cos\psi,\ -a\omega\sin\psi),\\
&f_1(a,\ \psi,\ \nu t) = f'_x(\nu t,\ a\cos\psi,\ -a\omega\sin\psi)\,u_1 +\\
&+ f'_{x'}(\nu t,\ a\cos\psi,\ -a\omega\sin\psi)\left[A_1\cos\psi - aB_1\sin\psi + \frac{\partial u_1}{\partial\psi}\omega +\right.\\
&\left. + \frac{\partial u_1}{\partial t}\right] + \left(aB_1^2 - \frac{dA_1}{da}A_1\right)\cos\psi + \left(\frac{dB_1}{da}A_1 a + 2A_1B_1\right)\sin\psi\\
&\quad - 2\omega B_1\frac{\partial^2 u_1}{\partial\psi^2} - 2\frac{\partial^2 u_1}{\partial a\,\partial t}A_1 - 2\frac{\partial^2 u_1}{\partial\psi\,\partial t}B_1 - 2\frac{\partial^2 u_1}{\partial a\,\partial\psi}\omega A_1,
\end{aligned}
\tag{12.15}
$$

. . . .

. . . .

It is obvious that the functions $f_k(a,\ \psi,\ \nu t)$ are periodic functions (with the period 2π) of both arguments ψ and νt and that, in addition, they depend on a. The explicit expression for these functions will be known as soon as we find the values of $A_j(a)$, $B_j(a)$, $u_j(a,\ \psi,\ \nu t)$ $(j = 1, 2, \ldots, k)$.

Before passing to a determination of the functions in which we are interested, we will briefly present certain aspects of the theory of multiple Fourier series.

If $f(x)$ is a certain periodic function of x, having the period 2π (in the case of the arbitrary period $2l$ we will always be able, by means of a linear transformation on x, to reduce it to the period 2π), then, as is commonly known, it may be represented, with certain restrictions, in the form of a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n\cos nx + b_n\sin nx\}, \tag{12.16}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\cos n\xi\,d\xi,\quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\sin n\xi\,d\xi. \tag{12.17}$$

In many cases, it is more convenient to use a Fourier series in the complex form.

In this case, $f(x)$ may be represented in the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{12.18}$$

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where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\, e^{-in\xi}\, d\xi. \tag{12.19}$$

(Here the subscript assumes not only values of positive integers but also negative values). In this case we obtain the following relation between the Fourier coefficients of eqs.(12.19) and (12.17):

$$c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}. \tag{12.20}$$

Let us assume that the function f(x,y) is periodic, having a period 2π with respect to both variables x and y.

On formal consideration of f(x,y) as a function of x, we have:

$$f(x,\ y) = \sum_{n=-\infty}^{\infty} c_n(y)\, e^{inx}, \tag{12.21}$$

where

$$c_n(y) = \frac{1}{2\pi}\int_0^{2\pi} f(\xi,\ y)\, e^{-in\xi}\, d\xi. \tag{12.22}$$

The function $c_n(y)$ may, in turn, be expanded into a series of the form

$$c_n(y) = \sum_{m=-\infty}^{\infty} c_{nm} e^{imy}, \tag{12.23}$$

where

$$\begin{aligned} c_{nm} &= \frac{1}{2\pi}\int_0^{2\pi} c_n(\eta)\, e^{-im\eta}\, d\eta = \\ &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(\xi,\ \eta)\, e^{-i(n\xi+m\eta)}\, d\xi\, d\eta. \end{aligned} \tag{12.24}$$

On substituting the resultant expression for $c_n(y)$ in eq.(12.21), we have

$$f(x,\ y) = \sum_{n=-\infty}^{\infty}\ \sum_{m=-\infty}^{\infty} c_{nm} e^{i(nx+my)}, \tag{12.25}$$

or, in abbreviated notation, the following formula

$$f(x,\ y) = \sum_{n,\ m=-\infty}^{\infty} c_{nm} e^{i(nx+my)}, \tag{12.26}$$

which generalizes the Fourier series to the case of two variables.

In the same way, the periodic function $f(x_1, x_2, \ldots, x_N)$, in N independent variables, with a period of 2π with respect to each of these variables, will read

$$f(x_1, x_2, \ldots, x_N) = \sum_{n_1, n_2, \ldots, n_N = -\infty}^{\infty} c_{n_1 n_2 \ldots n_N} e^{i(n_1 x_1 + n_2 x_2 + \ldots + n_N x_N)}, \tag{12.27}$$

where

$$c_{n_1 n_2 \ldots n_N} = \frac{1}{(2\pi)^N} \int_0^{2\pi} \int_0^{2\pi} \ldots$$

$$\ldots \int_0^{2\pi} f(\xi_1, \xi_2, \ldots, \xi_N) e^{-i(n_1\xi_1 + n_2\xi_2 + \ldots + n_N\xi_N)} d\xi_1 \, d\xi_2 \ldots d\xi_N. \tag{12.28}$$

This complex-exponential form of the multiple Fourier series is very convenient for calculations. It must, however be emphasized that it is completely equivalent to the ordinary form of expansion in sines and cosines, so that the conditions of convergence will be the same.

We will then proceed to the determination of $A_1(a)$, $B_1(a)$ and $u_1(a, \psi, \nu t)$ from eq.(12.13). For this purpose, let us expand $f_0(a, \psi, \nu t)$ into the double Fourier series

$$f_0(a, \psi, \nu t) = \sum_n \sum_m f_{nm}^{(0)}(a) e^{i(n\nu t + m\psi)}, \tag{12.29}$$

where

$$f_{nm}^{(0)}(a) = \frac{\nu}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(\nu t, \ a\cos\psi, \ -a\omega\sin\psi) \, e^{-i(n\nu t + m\psi)} dt \, d\psi.$$

Let us represent $u_1(a, \psi, \nu t)$ in the form of a Fourier series

$$u_1(a, \psi, \nu t) = \sum_n \sum_m \bar{f}_{nm}(a) e^{i(n\nu t + m\psi)}. \tag{12.30}$$

Substituting in eq.(12.13) the value of $f_0(a, \psi, \nu t)$ from eq.(12.29) and the value of $u_1(a, \psi, \nu t)$ from eq.(12.30), we have

$$\sum_n \sum_m \{\omega^2 - (n\nu + m\omega)^2\} \bar{f}_{nm}(a) e^{i(n\nu t + m\psi)} =$$

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$$= 2a\omega B_1 \cos\psi + 2\omega A_1 \sin\psi + \sum_n \sum_m f^{(0)}_{nm}(a)\, e^{i(n\nu t + m\psi)}. \tag{12.31}$$

From eq.(12.31) we must determine $\bar{f}_{nm}(a)$, $A_1(a)$ and $B_1(a)$ in such a way that $u_1(a, \psi, \nu t)$ will not contain resonant terms. The latter condition will be satisfied if $A_1(a)$ and $B_1(a)$ are determined from the relation

$$2a\omega B_1 \cos\psi + 2\omega A_1 \sin\psi = \sum_{\substack{n \quad m \\ [\omega^2 - (n\nu + m\omega)^2 = 0]}} \sum f^{(0)}_{nm}(a)\, e^{i(n\nu t + m\psi)}. \tag{12.32}$$

On equating the coefficients of the same harmonics in eq.(12.31), we get

$$\bar{f}_{nm}(a) = \frac{f^{(0)}_{nm}(a)}{\omega^2 - (n\nu + m\omega)^2}$$

for all values of n and m satisfying the inequality

$$\omega^2 - (n\nu + m\omega)^2 \neq 0,$$

or, in view of the fact that we are considering the nonresonant case, the inequality

$$n^2 + (m^2 - 1)^2 \neq 0 \quad (\text{т. е. } n \neq 0,\ m \neq \pm 1).$$

On substituting the value of $\bar{f}_{nm}(a)$ so found in eq.(12.30), and, substituting $\nu t = \theta$ for simplification, we find the following expression for $u_1(a, \psi, \nu t)$:

$$u_1(a, \psi, \theta) = \frac{1}{4\pi^2} \sum_{\substack{n \quad m \\ [n^2 + (m^2-1)^2 \neq 0]}} \sum \frac{e^{i(n\theta + m\psi)}}{\omega^2 - (n\nu + m\omega)^2} \times$$
$$\times \int_0^{2\pi} \int_0^{2\pi} f_0(a, \psi, \theta)\, e^{-i(n\theta + m\psi)} d\theta\, d\psi \tag{12.33}$$

or, passing to trigonometric functions,

$$u_1(a, \psi, \theta) = \frac{1}{2\pi^2} \sum_{\substack{n, m \\ [n^2 + (m^2-1)^2 \neq 0]}} \left\{ \frac{\cos(n\theta + m\psi)}{\omega^2 - (n\nu + m\omega)^2} \times \right.$$
$$\times \int_0^{2\pi} \int_0^{2\pi} f_0(a, \psi, \theta) \cos(n\theta + m\psi)\, d\theta\, d\psi +$$

$$+\frac{\sin(n\theta+m\psi)}{\omega^2-(n\nu+m\omega)^2}\int_0^{2\pi}\int_0^{2\pi} f_0(a,\psi,\theta)\sin(n\theta+m\psi)\,d\theta\,d\psi\Big\}. \quad (12.34)$$

On equating the coefficients of harmonics of the same degree in eq.(12.32), we find the following expression for $A_1(a)$ and $B_1(a)$*:

$$\left.\begin{aligned} A_1(a) &= -\frac{1}{4\pi^2\omega}\int_0^{2\pi}\int_0^{2\pi} f_0(a,\ \psi,\ \theta)\sin\psi\,d\theta\,d\psi,\\ B_1(a) &= -\frac{1}{4\pi^2\omega a}\int_0^{2\pi}\int_0^{2\pi} f_0(a,\ \psi,\ \theta)\cos\psi\,d\theta\psi. \end{aligned}\right\} \quad (12.35)$$

After determining $u_1(a,\ \psi,\ \theta)$, $A_1(a)$ and $B_1(a)$ we have, in accordance with equation (12.28), an explicit expression for $f_1(a,\ \psi,\ \theta)$. On expanding into a Fourier series and making use of eq.(12.14), and also taking account of the condition that resonant terms must be absent from the expression for $u_2(a,\ \psi,\ \theta)$, we now find by analogy $u_2(a,\ \psi,\ \theta)$, $A_2(a)$, and $B_2(a)$, which are necessary for constructing the second approximation. After a series of calculations we get

$$\begin{aligned} u_2(a,\ \psi,\ \theta) = \frac{1}{4\pi^2}\sum_{n}\sum_{m}{}_{[n^2+(m^2-1)^2\neq 0]} &\frac{e^{i(n\theta+m\psi)}}{\omega^2-(n\nu+m\omega)^2}\times\\ &\times\int_0^{2\pi}\int_0^{2\pi} f_1(a,\ \psi,\ \theta)\,e^{-i(n\theta+m\psi)}\,d\theta\,d\psi, \end{aligned} \quad (12.36)$$

$$\begin{aligned} A_2(a) = -\frac{1}{2\omega}\left[\frac{dB_1}{da}aA_1+2A_1B_1\right]&-\\ -\frac{1}{4\pi^2\omega}\int_0^{2\pi}\int_0^{2\pi}&\Big\{f'_x(\theta,\ a\cos\psi,\ -a\omega\sin\psi)\,u_1+\\ &+f'_{x'}(\theta,\ a\cos\psi,\ -a\omega\sin\psi)\Big(A_1\cos\psi-\\ &-aB_1\sin\psi+\frac{\partial u_1}{\partial\psi}\omega+\frac{\partial u_1}{\partial\theta}\nu\Big)\Big\}\sin\psi\,d\theta\,d\psi, \end{aligned}$$

* The right side of eq.(12.32), as easily demonstrated, contains only the first harmonic of the angle ψ.

$$
\left.
\begin{aligned}
B_2(a) = {} & \frac{1}{2a\omega}\left[\frac{dA_1}{dA}A_1 - aB_1^2\right] - \\
& - \frac{1}{4\pi^2\omega a}\int_0^{2\pi}\int_0^{2\pi}\Big\{f'_x(\theta,\ a\cos\psi,\ -a\omega\sin\psi)u_1 + \\
& + f'_{x'}(\theta, a\cos\psi, -a\omega\sin\psi)\Big(A_1\cos\psi - \\
& - aB_1\sin\psi + \frac{\partial u_1}{\partial\psi}\omega + \frac{\partial u_1}{\partial\theta}\nu\Big)\Big\}\cos\psi\, d\theta\, d\psi.
\end{aligned}
\right\} \qquad (12.37)
$$

Continuing this process of successive determination of the expressions in which we are interested, we may construct the solution of eq.(12.1) to any desired approximation.

We note that, starting from arguments analogous to those given in Chapter 1, it is again of no use, in constructing the n^{th} approximation, to retain in the right-hand side of the series (12.7) a term of the order of smallness of ϵ^n.

In concluding our discussion of the nonresonant case, we remark that, according to eq.(12.35), the equation of first approximation contains only the free term $f_0(x, \frac{dx}{dt})$ of the expansion (12.2) of the perturbation $f(\theta, x, \frac{dx}{dt})$.

On the basis of eq.(12.2) we have, identically,

$$
f_0\left(x,\ \frac{dx}{dt}\right) = \lim_{T\to\infty}\frac{1}{T}\int_0^T f\left(\tau,\ x,\ \frac{dx}{dt}\right)d\tau. \qquad (12.38)
$$

For this reason, the equation of first approximation can be obtained by averaging the disturbing force over the time explicitly contained in it, after which we may use the formulas of Section 1, eq.(1.27).

In the nonresonant case under consideration, the equations of first approximation (and also those of higher approximations) have the same form as the equations of first approximation for the case of eq.(1.1) (i.e., for the case when the external disturbing forces do not depend explicitly on the time), which have already been discussed by us in detail, and therefore will be disregarded here.

We will discuss only the expression for x in second approximation

$$x = a\cos\psi + \varepsilon u_1(a, \psi, \vartheta), \tag{12.39}$$

where $u_1(a, \psi, \theta)$ is defined by eq.(12.33) or (12.34).

In the nonresonant case, according to eqs.(12.39) and (12.36), the influence of the external periodic action manifests itself only in second approximation. For example, it follows directly from eq.(12.39) that only in the second approximation can various compound harmonics, components with various multiple frequencies of the constraining force, etc. appear in the solution. The amplitude of all these additional harmonics will be of the same order of smallness as ε.

Let us consider eq.(12.39) in the case of stationary oscillations

$$a = \text{const}, \quad \psi = \omega(a)t + \vartheta, \quad \vartheta = \text{const}. \tag{12.40}$$

In this case, the oscillating term x consists of a natural oscillation of the frequency $\omega(a)$ (represented by the term $a\cos[\omega(a)t + \theta]$), forced oscillations with the frequencies $n\nu(n = 1, 2, 3, \ldots)$ and of compound oscillations with the frequencies $n\nu \pm m\omega(n,m = 1, 2, 3, \ldots)$. In this case, the intensity of the compound oscillation with the frequency $n\nu \pm m\omega$ is increased as the corresponding resonance is approached, i.e., as the corresponding divisor

$$\omega^2 - (n\nu \pm m\omega)^2.$$

is decreased.

In the special case where there are no natural oscillations, i.e., when $a = 0$, eq.(12.39) degenerates into

$$x = \varepsilon \sum_{n=1}^{\infty} \frac{A_n \cos n\vartheta + B_n \sin n\vartheta}{\omega^2 - n^2\nu^2}, \tag{12.41}$$

where the following notation is used:

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f_0(\vartheta, 0, 0)\cos n\vartheta\, d\vartheta,$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f_0(\vartheta, 0, 0)\sin n\vartheta\, d\vartheta.$$

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Thus, when $a = 0$ in an oscillatory system, there are only certain forced vibrations with frequencies of the external excitation $n\nu$ ($n = 1, 2, 3, \ldots$).

For this reason, we have here to do with purely forced oscillations. The states of oscillation corresponding to eq.(12.41) are sometimes termed heteroperiodic, since the periods of all harmonics of the oscillation are imposed on the system from outside.

If the oscillatory system under study is such that the equivalent damping decrement for the component $\varepsilon f_0(x, \frac{dx}{dt})$ (which does not depend explicitly on the time) of the disturbing function $\varepsilon f(\theta, x, \frac{dx}{dt})$ is positive

$$\lambda_e^*(a) > 0,$$

$$\lambda_e^*(a) = \frac{1}{4\pi^2\omega} \int_0^{2\pi} \int_0^{2\pi} f(\theta, a\cos\psi, -a\omega\sin\psi)\sin\psi \, d\theta \, d\psi = \tag{12.42}$$

$$= \frac{1}{2\pi\omega} \int_0^{2\pi} f_0(a\cos\psi, -a\omega\sin\psi)\sin\psi \, d\psi,$$

then (cf.Section 7)

$$\underset{t\to\infty}{a(t)} \to 0,$$

Therefore, if the inequality (12.42) is satisfied, every oscillation will approach a heteroperiodic oscillation, so that the heteroperiodic state will be the only possible stationary state.

The condition (12.42) obtained for damping the natural oscillation, generally speaking, will depend on the amplitude of the external periodic force. In the absence of external excitation, i.e., in the case where the right side of eq.(12.1) does not depend explicitly on the time, we obtain the ordinary condition for self-excitation

$$\lambda_e(a) < 0 \tag{12.43}$$

and, correspondingly, the condition of damping

$$\lambda_e(a) > 0, \tag{12.44}$$

where

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$$\lambda_e(a) = \frac{1}{2\pi\omega} \int_0^{2\pi} f(0,\ a\cos\psi,\ -a\omega\sin\psi)\sin\psi\, d\psi\ ^{*)}.$$

Depending on the structure of the nonlinear function $f(\theta,\ x,\ \frac{dx}{dt})$, it may happen that the conditions (12.43) and (12.42) are satisfied at the same time. In that case, it will be found that the system which is self-excited in the absence of an external periodic influence, loses the self-excitation in the presence of an external periodic influence. In this case we have to do with nonresonant or asynchronous extinction.

By analogy, the opposite case of synchronous excitation may also be presented.

At the beginning of the preceding Section we assumed that the right side of the differential equation (12.1) under study $f(\nu t,\ x,\ \frac{dx}{dt})$ was a periodic function in t with the period $\frac{2\pi}{\nu}$, and that it might also be represented in the form of the finite sum (12.2), in which the coefficients $f_n(x,\ \frac{dx}{dt})$ are certain polynomials in x and $\frac{dx}{dt}$.

If we make a more general assumption and postulate that the function $f(\nu t,\ x,\ \frac{dx}{dt})$ may be represented in the form of a uniformly convergent series

$$f\left(\nu t,\ x,\ \frac{dx}{dt}\right) = \sum_{n=-\infty}^{\infty} e^{in\nu t} f_n\left(x,\ \frac{dx}{dt}\right), \qquad (12.45)$$

in which $f_n(x,\ \frac{dx}{dt})$ are certain arbitrary regular functions of x and $\frac{dx}{dt}$, then in the expression for $u_1(a,\ \psi,\ \theta)$, $u_2(a,\ \psi,\ \theta)$;..., the finite double sums will be replaced by double infinite series of the type

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{e^{i(n\theta+m\psi)}}{\omega^2-(n\nu+m\omega)^2} \int_0^{2\pi}\int_0^{2\pi} f_0(a,\ \psi,\ \theta)\, e^{-i(n\theta+m\psi)}\, d\theta\, d\psi. \qquad (12.46)$$

Because of the presence of a divisor of the form $\omega^2 - (n\nu + m\omega)^2$, these series, generally speaking, will be divergent.

It is well known that, in the general case, the branch points of series of this kind on the ν axis (i.e., values of ν at which the series diverges) form

---

* $f(0,\ a\cos\psi,\ -a\omega\sin\psi)$ denotes the disturbing force $f(\theta,\ x,\ \frac{dx}{dt})$, in which we put $x = a\cos\psi$, $\frac{dx}{dt} = -a\omega\sin\psi$, while the amplitude of the external periodic components is equal to zero.

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a network that is dense everywhere.

Thus, whatever the value of ν, a value ν_0 for which the series (12.46) is divergent may be found, as close as desired to that value.

On the other hand, we note that for almost all values of the ratio $\frac{\nu}{\omega}$ (with the possible exception of the set of zero measure), we may find* quantities C and δ such that

$$\left|\frac{\nu}{\omega}-\frac{p}{q}\right| \geqslant \frac{C}{(|p|+|q|)^{2+\delta}}$$

for any integers p, q.

However, in this case,

$$|n\nu+(m\pm 1)\omega| \geqslant \frac{C\omega}{n^{\delta+1}}.$$

and, in absolute value, each term of the series (12.46) will be, respectively,

---

* Indeed, let us fix a certain positive δ and positive η, as small as desired. Let us take a positive value of C such that

$$2C\sum_{|n|+|m|\geq 1}\frac{1}{(|n|+|m|)^{2+\delta}}\leqslant \eta$$

and let us construct the set of intervals l_{mn} (where m and n are any desired positive and negative integers) with centers at the points $\frac{n}{m}$ and lengths $\frac{2C}{(|n|+|m|)^{2+\delta}}$

On the one hand, it is clear that for any number x not belonging to even a single one of the intervals $l_{n,m}$, the inequality

$$\left|x-\frac{n}{m}\right| \geqslant \frac{C}{(|n|+|m|)^{2+\delta}}. \qquad (\alpha)$$

is satisfied for any integers n,m.

On the other hand, the set x, which does belong to one of the intervals $l_{n,m}$ has a measure smaller than

$$\sum_{n,m}\operatorname{mes} l_{n,m}\leqslant\eta.$$

Thus, for all values of x, with the possible exception of the x belonging to a set of measure smaller than η, the inequality (α) will be satisfied.

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smaller than

$$\frac{n^{s+1}}{C\omega}\left|\int_0^{2\pi}\int_0^{2\pi} f_0(a,\psi,\theta)\,e^{-i(n\theta+m\psi)}\,d\theta\,d\psi\right|.$$

This series will thus be absolutely convergent provided only that $f_o(a, \psi, \theta)$ possesses a sufficient number of continuous partial derivatives with respect to the angular variables ψ, θ.

However, in order to avoid going into such refinements of the theory of numbers, it is preferable in practical applications not to go as far as the appearance of infinite sums of harmonic summands and to refer the residue of the series to higher powers of ε.

In other words, it is more convenient to start from equations of the type

$$\frac{d^2x}{dt^2}+\omega^2x=\varepsilon f_0\left(\nu t,\ x,\ \frac{dx}{dt}\right)+\varepsilon^2 f_1\left(\nu t,\ x,\ \frac{dx}{dt}\right)+\varepsilon^3\ldots, \tag{12.47}$$

in which $f_o(\nu t, x, \frac{dx}{dt})$, $f_1(\nu t, x, \frac{dx}{dt})$,.... are already finite sums of the type of eq.(12.2).

The extension of the above technique of constructing approximate solutions, to apply to eq.(12.47), involves no difficulties.

Let us now discuss a concrete example.

Consider the generalized van der Pol equation

$$\frac{d^2y}{dt^2}+y-\varepsilon(1-y^2)\frac{dy}{dt}=E\sin\nu t, \tag{12.48}$$

where ε is a certain small positive parameter.

As pointed out above, a substitution of the type of

$$y=x+U\sin\nu t, \tag{12.49}$$

where $U = \frac{E}{1-\nu^2}$ (we are considering the nonresonant case, so that ν is not equal to unity nor close to it), this equation is brought into the form

$$\frac{d^2x}{dt^2}+x=\varepsilon\left(1-(x+U\sin\nu t)^2\right)\left(\frac{dx}{dt}+U\nu\cos\nu t\right). \tag{12.50}$$

Making use of eq.(12.35), we obtain the solution of eq.(12.50) in first ap-

proximation,

$$x = a\cos(t+\theta), \tag{12.51}$$

where θ = const, while a must obviously be determined from the equation

$$\frac{da}{dt} = \frac{\varepsilon a}{2}\left(1 - \frac{a^2}{4} - \frac{U^2}{2}\right). \tag{12.52}$$

The equation of first approximation (12.52) shows that, for

$$U^2 < 2 \tag{12.53}$$

the system is self-exciting, and that there exists a stable stationary state of oscillations corresponding to the amplitude

$$a^2 = 4 - 2U^2.$$

For

$$U^2 > 2 \tag{12.54}$$

the amplitude a, with increasing t, tends toward zero so that asynchronous extinction takes place in the system.

Let us now find the solution of eq.(12.50) in second approximation. Making use of eqs.(12.39) and (12.37), we have

$$\begin{aligned} x = a\cos\psi &+ \frac{\varepsilon U\nu(4 - U^2 - 2a^2)}{4(1-\nu^2)}\cos\theta + \frac{\varepsilon U^3\nu}{4(1-9\nu^2)}\cos 3\theta + \\ &+ \frac{\varepsilon U a^2(2+\nu)}{4(1+\nu)(3+\nu)}\cos(\theta + 2\psi) + \frac{\varepsilon U a^2(2-\nu)}{4(1-\nu)(3-\nu)} \times \\ &\times \cos(\theta - 2\psi) + \frac{\varepsilon U^2 a(2+\nu)}{16\nu(1+\nu)}\sin(2\theta + \psi) + \\ &+ \frac{\varepsilon U^2 a(1-2\nu)}{16\nu(1-\nu)}\sin(2\theta - \psi) - \frac{a^3}{32}\sin 3\psi, \end{aligned} \tag{12.55}$$

where a and ψ must satisfy the system of second approximation

$$\left.\begin{aligned} \frac{da}{dt} &= \frac{\varepsilon a}{2}\left(1 - \frac{a^2}{4} - \frac{U^2}{2}\right), \\ \frac{d\psi}{dt} &= 1 - \varepsilon^2\left(\frac{1}{8} - \frac{a^2}{8} + \frac{7a^4}{256}\right) + \frac{\varepsilon^2 U^2(5\nu - 1)}{8(1-\nu^2)} + \\ &+ \frac{\varepsilon^2 U^2 a^2(7\nu^4 - 40\nu^2 + 32\nu - 9)}{32(9-\nu^2)(1-\nu^2)} + \frac{\varepsilon^2 U^4(1 + 4\nu - 8\nu^2)}{64(1-\nu^2)}. \end{aligned}\right\} \tag{12.56}$$

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As was to be expected, in our second approximation, the forced oscillations with frequencies of ν and 3ν, equal to the frequencies of the external force, are accompanied by components with multiple frequencies 3ω and with compound frequencies $\nu \pm 2\omega$, $2\nu \pm \omega$, which are characteristic only of nonlinear systems.

Moreover, on the basis of the above statements in satisfying the condition (12.53), the heteroperiodic oscillation state is unstable and thus physically impossible. In the case where it satisfies the condition (12.54), a heteroperiodic state will be the only stable stationary state. With the passage of time, heteroperiodic oscillations of the form

$$x = \frac{\varepsilon U\nu(4-U^2)}{4(1-\nu^2)}\cos\theta + \frac{\varepsilon U^3\nu}{4(1-9\nu^2)}\cos 3\theta. \qquad (12.57)$$

will become established in the system.

Section 13. The "Resonant" Cases

Let us now discuss the resonant cases.

Assume that

$$\omega \approx \frac{p}{q}\nu,$$

where p and q are certain relatively prime numbers.

Then, depending on the character of the problem involved, two different approaches to its solution may arise: 1) in investigating the resonance it is sufficient to confine the consideration to the resonant region itself; 2) besides studying the resonant region, it is also necessary to study the approaches to this region from the nonresonant zone.

Let us start the discussion with the former case, since it is simpler. In view of the fact that, in this case, we assume that we are considering values of $\frac{p}{q}\nu$ sufficiently close to ω, it is natural to put

$$\omega^2 = \left(\frac{p}{q}\nu\right)^2 + \varepsilon\Delta, \qquad (13.1)$$

where $\varepsilon\Delta$ represents the detuning between the squares of the natural and external frequencies.

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Then the initial eq.(12.1) will be written in the form

$$\frac{d^2x}{dt^2}+\left(\frac{p}{q}\nu\right)^2 x=\varepsilon\left\{f\left(\nu t,\ x,\ \frac{dx}{dt}\right)-\Delta x\right\}. \qquad (13.2)$$

Thus the "detuning" $\varepsilon\Delta$, in view of its smallness, will be related to the perturbation, after which the solution of eq.(13.2), as in the nonresonant case, can be sought in the form

$$x=a\cos\psi+\varepsilon u_1(a,\ \psi,\ \nu t)+\varepsilon^2 u_2(a,\ \psi,\ \nu t)+\ldots, \qquad (13.3)$$

where a and ψ are certain functions of time. Here we have $\psi = \frac{p}{q}\nu t + \theta$; θ = phase detuning (at exact resonance, θ = const).

In the cases previously considered, the instantaneous frequency and amplitude of the oscillations did not depend on the phase difference; however in the resonant case, as shown above, the phase difference will have a substantial influence on both amplitude and frequency of the oscillations, so that we are forced to introduce the dependence on the phase detuning

$$\theta=\psi-\frac{p}{q}\nu t.$$

in the right sides of the equation determining a and ψ.

Thus, in eq.(13.3) we must substitute a and ψ as functions of time, determined from the following system of equation:

$$\left.\begin{aligned}\frac{da}{dt}&=\varepsilon A_1(a,\ \theta)+\varepsilon^2 A_2(a,\ \theta)+\ldots,\\ \frac{d\psi}{dt}&=\frac{p}{q}\nu+\varepsilon B_1(a,\ \theta)+\varepsilon^2 B_2(a,\ \theta)+\ldots\end{aligned}\right\} \qquad (13.4)$$

Since not the full phase ψ but the phase angle θ enters the right sides of the expressions for $\frac{da}{dt}$ and $\frac{d\psi}{dt}$ of eq.(13.4), it will be expedient to eliminate ψ from eqs.(13.3) and (13.4).

Then, putting $\psi = \frac{p}{q}\nu t + \theta$, we obtain the following expression instead of equation (13.3):

$$x=a\cos\left(\frac{p}{q}\nu t+\theta\right)+\varepsilon u_1\left(a,\ \theta,\ \frac{\nu}{q}t\right)+\varepsilon^2 u_2\left(a,\ \theta,\ \frac{\nu}{q}t\right)+\ldots, \qquad (13.5)$$

in which the functions of time a and θ must satisfy the equation

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$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a,\ \theta) + \varepsilon^2 A_1(a,\ \theta) + \ldots, \\ \frac{d\theta}{dt} &= \varepsilon B_1(a,\ \theta) + \varepsilon^2 B_2(a,\ \theta) + \ldots \end{aligned}\right\} \tag{13.6}$$

As in the preceding case, for the determination of the functions $u_1(a,\ \theta,\ \frac{\nu}{q}\ t)$, $u_2(a,\ \theta,\ \frac{\nu}{q}\ t), \ldots,\ A_1(a,\ \theta),\ A_2(a,\ \theta), \ldots,\ B_1(a,\ \theta),\ B_2(a,\ \theta), \ldots$ we find at first

$$\begin{aligned} \frac{dx}{dt} = &\left\{\cos\left(\frac{p}{q}\nu t + \theta\right) + \varepsilon\frac{\partial u_1}{\partial a} + \varepsilon^2\frac{\partial u_2}{\partial a} + \ldots\right\}\frac{da}{dt} + \\ &+\left\{-a\sin\left(\frac{p}{q}\nu t + \theta\right) + \varepsilon\frac{\partial u_1}{\partial \theta} + \varepsilon^2\frac{\partial u_2}{\partial \theta} + \ldots\right\}\frac{d\theta}{dt} + \\ &+\left\{-a\,\frac{p}{q}\nu\sin\left(\frac{p}{q}\nu t + \theta\right) + \varepsilon\frac{\partial u_1}{\partial t} + \varepsilon^2\frac{\partial u_2}{\partial t} + \ldots\right\}, \end{aligned} \tag{13.7}$$

$$\begin{aligned} \frac{d^2x}{dt^2} = &\left\{\cos\left(\frac{p}{q}\nu t + \theta\right) + \varepsilon\frac{\partial u_1}{\partial a} + \varepsilon^2\frac{\partial u_2}{\partial a} + \ldots\right\}\frac{d^2a}{dt^2} + \\ &+\left\{\varepsilon\frac{\partial^2 u_1}{\partial a^2} + \varepsilon^2\frac{\partial^2 u_2}{\partial a^2} + \ldots\right\}\left(\frac{da}{dt}\right)^2 + \\ &+2\left\{-\sin\left(\frac{p}{q}\nu t + \theta\right) + \varepsilon\frac{\partial^2 u_1}{\partial a\,\partial\theta} + \varepsilon^2\frac{\partial^2 u_2}{\partial a\,\partial\theta} + \ldots\right\}\frac{da}{dt}\frac{d\theta}{dt} + \\ &+2\left\{-\frac{p}{q}\nu\sin\left(\frac{p}{q}\nu t + \theta\right) + \varepsilon\frac{\partial^2 u_1}{\partial a\,\partial t} + \varepsilon^2\frac{\partial^2 u_2}{\partial a\,\partial t} + \ldots\right\}\frac{da}{dt} + \\ &+\left\{-a\sin\left(\frac{p}{q}\nu t + \theta\right) + \varepsilon\frac{\partial u_1}{\partial\theta} + \varepsilon^2\frac{\partial u_2}{\partial\theta} + \ldots\right\}\frac{d^2\theta}{dt^2} + \\ &+\left\{-a\cos\left(\frac{p}{q}\nu t + \theta\right) + \varepsilon\frac{\partial^2 u_1}{\partial\theta^2} + \varepsilon^2\frac{\partial^2 u_2}{\partial\theta^2} + \ldots\right\}\left(\frac{d\theta}{dt}\right)^2 + \\ &+2\left\{-a\,\frac{p}{q}\nu\cos\left(\frac{p}{q}\nu t + \theta\right) + \varepsilon\frac{\partial^2 u_1}{\partial\theta\,\partial t} + \varepsilon^2\frac{\partial^2 u_2}{\partial\theta\,\partial t} + \ldots\right\}\frac{d\theta}{dt} + \\ &+\left\{-a\left(\frac{p}{q}\nu\right)^2\cos\left(\frac{p}{q}\nu t + \theta\right) + \varepsilon\frac{\partial^2 u_1}{\partial t^2} + \varepsilon^2\frac{\partial^2 u_2}{\partial t^2} + \ldots\right\}. \end{aligned} \tag{13.8}$$

Then, on the basis of eq.(13.6), we have

$$\left.\begin{aligned} &\frac{d^2a}{dt^2} = \varepsilon^2\left\{\frac{\partial A_1}{\partial a}A_1 + \frac{\partial A_1}{\partial\theta}B_1\right\} + \varepsilon^3\ldots, \\ &\left(\frac{da}{dt}\right)^2 = \varepsilon^2 A_1^2 + \varepsilon^3\ldots,\ \left(\frac{d\theta}{dt}\right)^2 = \varepsilon^2 B_1^2 + \varepsilon^3\ldots, \end{aligned}\right\} \tag{13.9}$$

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$$\frac{d^2\vartheta}{dt^2} = \varepsilon^2\left\{\frac{\partial B_1}{\partial a}A_1 + \frac{\partial B_1}{\partial \vartheta}B_1\right\} + \varepsilon^3\ldots, \qquad \frac{da}{dt}\,\frac{d\vartheta}{dt} = \varepsilon^2 A_1 B_1 + \varepsilon^3\ldots$$

On substituting eqs.(13.5) and (13.8) in the left side of eq.(13.2), taking account in this case of eqs.(13.6) and (13.9), and arranging the result by powers of the parameter ε, we get

$$\frac{d^2x}{dt^2} + \left(\frac{p}{q}\nu\right)^2 x = \varepsilon\left\{-2\frac{p}{q}\nu\sin\left(\frac{p}{q}\nu t + \vartheta\right)A_1 - \right.$$

$$\left. - 2a\frac{p}{q}\nu\cos\left(\frac{p}{q}\nu t + \vartheta\right)B_1 + \frac{\partial^2 u_1}{\partial t^2} + \left(\frac{p}{q}\nu\right)^2 u_1\right\} +$$

$$+ \varepsilon^2\left\{\left[\frac{\partial A_1}{\partial a}A_1 + \frac{\partial A_1}{\partial \vartheta}B_1 - aB_1^2 - \right.\right.$$

$$\left. - 2a\frac{p}{q}\nu B_2\right]\cos\left(\frac{p}{q}\nu t + \vartheta\right) + \left[-2A_1B_1 - 2\frac{p}{q}\nu A_2 - \right.$$

$$\left. - a\frac{\partial B_1}{\partial a}A_1 - a\frac{\partial B_1}{\partial \vartheta}B_1\right]\sin\left(\frac{p}{q}\nu t + \vartheta\right) + 2\frac{\partial^2 u_1}{\partial a\,\partial t}A_1 +$$

$$\left. + 2\frac{\partial^2 u_1}{\partial \vartheta\,\partial t}B_1 + \frac{\partial^2 u_2}{\partial t^2} + \left(\frac{p}{q}\nu\right)^2 u_2\right\} + \varepsilon^3\ldots \qquad (13.10)$$

Expanding the right side of eq.(13.2) in powers of the small parameter, we find

$$\varepsilon\left\{f\left(\nu t,\ x,\ \frac{dx}{dt}\right) - \Delta x\right\} = \varepsilon\left\{-\Delta\cdot a\cos\left(\frac{p}{q}\nu t + \vartheta\right) + \right.$$

$$\left. + f\left(\nu t,\ a\cos\left(\frac{p}{q}\nu t + \vartheta\right),\ -a\frac{p}{q}\nu\sin\left(\frac{p}{q}\nu t + \vartheta\right)\right)\right\} +$$

$$+ \varepsilon^2\left\{f'_x\left(\nu t, a\cos\left(\frac{p}{q}\nu t + \vartheta\right), -a\frac{p}{q}\nu\sin\left(\frac{p}{q}\nu t + \vartheta\right)\right)u_1 + \right.$$

$$+ f'_{x'}\left(\nu t,\ a\cos\left(\frac{p}{q}\nu t + \vartheta\right),\ -a\frac{p}{q}\nu\sin\left(\frac{p}{q}\nu t + \vartheta\right)\right)\times$$

$$\times\left(A_1\cos\left(\frac{p}{q}\nu t + \vartheta\right) - aB_1\sin\left(\frac{p}{q}\nu t + \vartheta\right) + \frac{\partial u_1}{\partial t}\right) -$$

$$\left. - \Delta\cdot u_1\right\} + \varepsilon^3\ldots \qquad (13.11)$$

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Equating the coefficients of the same powers of ε in the right sides of equations (13.10) and (13.11), we obtain the following system of equations for determining the wanted functions

$$\frac{\partial^2 u_1}{\partial t^2} + \left(\frac{p}{q}\nu\right)^2 u_1 = f_0\left(a, \nu t, \frac{p}{q}\nu t + \theta\right) + 2\frac{p}{q}\nu A_1 \sin\left(\frac{p}{q}\nu t + \theta\right) + 2a\frac{p}{q}\nu B_1 \cos\left(\frac{p}{q}\nu t + \theta\right) - \Delta a \cos\left(\frac{p}{q}\nu t + \theta\right), \quad (13.12)$$

$$\frac{\partial^2 u_2}{\partial t^2} + \left(\frac{p}{q}\nu\right)^2 u_2 = f_1\left(a, \nu t, \frac{p}{q}\nu t + \theta\right) + \left[2\frac{p}{q}\nu A_2 + a\frac{\partial B_1}{\partial a}A_1 + a\frac{\partial B_1}{\partial \theta}B_1 + 2A_1B_1\right]\sin\left(\frac{p}{q}\nu t + \theta\right) + \left[2a\frac{p}{q}\nu B_2 - \frac{\partial A_1}{\partial a}A_1 - \frac{\partial A_1}{\partial \theta}B_1 + aB_1^2\right]\cos\left(\frac{p}{q}\nu t + \theta\right), \quad (13.13)$$

where the following notation has been introduced:

$$f_0\left(a, \nu t, \frac{p}{q}\nu t + \theta\right) = f\left(\nu t, a\cos\left(\frac{p}{q}\nu t + \theta\right), -a\frac{p}{q}\nu \sin\left(\frac{p}{q}\nu t + \theta\right)\right), \quad (13.14)$$

$$f_1\left(a, \nu t, \frac{p}{q}\nu t + \theta\right) = f'_x\left(\nu t, a\cos\left(\frac{p}{q}\nu t + \theta\right), -a\frac{p}{q}\nu \sin\left(\frac{p}{q}\nu t + \theta\right)\right)u_1 + f'_{x'}\left(\nu t, a\cos\left(\frac{p}{q}\nu t + \theta\right), -a\frac{p}{q}\nu \sin\left(\frac{p}{q}\nu t + \theta\right)\right)\times \left(A_1\cos\left(\frac{p}{q}\nu t + \theta\right) - aB_1\sin\left(\frac{p}{q}\nu t + \theta\right) + \frac{\partial u_1}{\partial t}\right) - \Delta u_1 - 2A_1\frac{\partial^2 u_1}{\partial a\,\partial t} - 2B_1\frac{\partial^2 u_1}{\partial \theta\,\partial t}. \quad (13.15)$$

As in the previously discussed cases, $f_k(a, \nu t, \frac{p}{q}\nu t + \theta)$ are periodic functions with the period 2π in both angular variables νt, $\frac{p}{q}\nu t + \theta$, [$A_i(a, \theta)$ and $B_i(a, \theta)$, as will be clear from the subsequent calculations, being periodic functions with re-

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spect to θ, with the period 2π].

Let us find from eq.(13.12) $u_1(a, \nu t, \frac{p}{q}\nu t + \theta)$, $A_1(a, \theta)$ and $B_1(a, \theta)$, observing the condition that terms whose denominators might vanish must be absent from the expressions for $u_1(a, \nu t, \frac{p}{q}\nu t + \theta)$.

On representing $u_1(a, \nu t, \frac{p}{q} + \theta)$, and also the right side of eq.(13.12) in the form of finite Fourier sums, we have

$$u_1\left(a, \nu t, \frac{p}{q}\nu t + \theta\right) = \sum_n \sum_m \overline{f_{nm}^{(1)}}(a)\, e^{i\left\{n\nu t + m\left(\frac{p}{q}\nu t + \theta\right)\right\}}, \quad (13.16)$$

$$f_0\left(a, \nu t, \frac{p}{q}\nu t + \theta\right) = \sum_n \sum_m f_{nm}^{(0)}(a)\, e^{i\left\{n\nu t + m\left(\frac{p}{q}\nu t + \theta\right)\right\}}. \quad (13.17)$$

Let us now substitute the right sides of eqs.(13.16) and (13.17) in eq.(13.12). This yields

$$\sum_n \sum_m \left\{\left(\frac{p}{q}\nu\right)^2 - \left(n\nu + m\frac{p}{q}\nu\right)^2\right\} e^{i\left\{n\nu t + m\left(\frac{p}{q}\nu t + \theta\right)\right\}}\, \overline{f_{nm}^{(1)}}(a) =$$

$$= \sum_n \sum_m f_{nm}^{(0)}(a)\, e^{i\left\{n\nu t + m\left(\frac{p}{q}\nu t + \theta\right)\right\}} + 2\frac{p}{q}\nu A_1 \sin\left(\frac{p}{q}\nu t + \theta\right) +$$

$$+ 2a\frac{p}{q}\nu B_1 \cos\left(\frac{p}{q}\nu t + \theta\right) - \Delta a \cos\left(\frac{p}{q}\nu t + \theta\right), \quad (13.18)$$

whence, by equating the coefficients for the same harmonics, we find

$$\overline{f_{nm}^{(1)}}(a) = \frac{f_{nm}^{(0)}(a)}{\left(\frac{p}{q}\nu\right)^2 - \left(n\nu + m\frac{p}{q}\nu\right)^2} \quad (13.19)$$

for all values of n and m satisfying the condition

$$\left(\frac{p}{q}\nu\right)^2 - \left(n\nu + m\frac{p}{q}\nu\right)^2 \neq 0,$$

or the equivalent condition

$$nq + (m \pm 1)p \neq 0;$$

We also obtain the relation for determining $A_1(a, \theta)$ and $B_1(a, \theta)$:

$$2\frac{p}{q}\nu A_1 \sin\left(\frac{p}{q}\nu t + \theta\right) + \left(2a\frac{p}{q}\nu B_1 - \Delta a\right)\cos\left(\frac{p}{q}\nu t + \theta\right) +$$

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$$+\sum_{\substack{n \\ [nq+(m\pm 1)p=0]}}\sum_{m} e^{i\left\{n\nu t+m\left(\frac{p}{q}\nu t+\theta\right)\right\}} f_{nm}^{(0)}(a)=0. \qquad (13.20)$$

On substituting the value of $\bar{f}_{nm}^{(1)}(a)$ from eq.(13.19) in the right side of equation (13.16), we find

$$u_1\left(a,\ \nu t,\ \frac{p}{q}\nu t+\theta\right)=\frac{1}{4\pi^2}\sum_{\substack{n \\ [nq+(m+1)p\neq 0]}}\sum_{m}\frac{e^{i\left\{n\nu t+m\left(\frac{p}{q}\nu t+\theta\right)\right\}}}{\left(\frac{p}{q}\nu\right)^2-\left(n\nu+m\frac{p}{q}\nu\right)^2}\times$$

$$\times\int_0^{2\pi}\int_0^{2\pi} f\left(a,\ \nu t,\ \frac{p}{q}\nu t+\theta\right)e^{-i\left\{n\nu t+m\left(\frac{p}{q}\nu t+\theta\right)\right\}}d\nu t\, d\left(\frac{p}{q}\nu t+\theta\right). \qquad (13.21)$$

Let us now turn to eq.(13.20). The summation in this case, as already indicated, proceeds for all integers n, m (positive, negative, and zero), for which

$$nq+(m\pm 1)p=0. \qquad (13.22)$$

For this reason, the sum will contain complex exponents of the form

$$e^{i\left\{\left(n+m\frac{p}{q}\right)\nu t+m\theta\right\}}=e^{i\left\{(nq+mp)\frac{\nu}{q}t+m\theta\right\}}=$$

$$=e^{i\left\{\mp\frac{p}{q}\nu t+m\theta\right\}}=e^{i\left\{\mp\left(\frac{p}{q}\nu t+\theta\right)+(m\pm 1)\theta\right\}}=$$

$$=\left\{\cos\left(\frac{p}{q}\nu t+\theta\right)\mp i\sin\left(\frac{p}{q}\nu t+\theta\right)\right\}e^{i(m\pm 1)\theta}.$$

We also note that, by virtue of eq.(13.22), $m \pm 1$ is divisible by q, so that we may write this factor in the form $q\sigma(-\infty < \sigma < \infty)$.

On equating the coefficients of $\cos(\frac{p}{q}\nu t+\theta)$ and $\sin(\frac{p}{q}\nu t+\theta)$ in eq.(13.20) we have

$$\left.\begin{aligned}
&A_1(a,\ \theta)=\frac{q}{4\pi^2\nu p}\sum_{\sigma}e^{iq\sigma\theta}\int_0^{2\pi}\int_0^{2\pi} f_0(a,\ \nu t,\ \psi)e^{-iq\sigma\theta}\sin\psi\, d\nu t\, d\psi,\\
&B_1(a,\ \theta)=\\
&=\frac{\Delta}{2}\frac{q}{p\nu}-\frac{q}{4\pi^2 ap\nu}\sum_{\sigma}e^{iq\sigma\theta}\int_0^{2\pi}\int_0^{2\pi} f_0(a,\ \nu t,\ \psi)e^{-iq\sigma\theta}\cos\psi\, d\nu t\, d\psi
\end{aligned}\right\} \qquad (13.23)$$

$$\left(\psi = \frac{p}{q}\nu t + \theta\right).$$

Thus, in first approximation for the resonant case, the solution of eq.(13.1) will be

$$x = a\cos\left(\frac{p}{q}\nu t + \theta\right),$$

where a and θ must be determined from the system of equations

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon q}{4\pi^2\nu p}\sum_{\sigma} e^{i q \sigma \theta}\int_0^{2\pi}\int_0^{2\pi} f_0(a, \nu t, \psi)\, e^{-i q \sigma \theta}\sin\psi\, d\nu t\, d\psi, \\ \frac{d\theta}{dt} &= \frac{\varepsilon\Delta q}{2p\nu} - \frac{\varepsilon q}{4\pi^2 a\nu p}\sum_{\sigma} e^{i q \sigma \theta}\int_0^{2\pi}\int_0^{2\pi} f_0(a, \nu t, \psi)\, e^{-i q \sigma \theta}\cos\psi\, d\nu t\, d\psi. \end{aligned}\right\} \quad (13.24)$$

Since, in the resonant case we assume the detuning εΔ to be a quantity of the first order of smallness, we may with the same degree of accuracy represent the system of eq.(13.24) in the form

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon q}{4\pi^2\nu p}\sum_{\sigma} e^{i q \sigma \theta}\int_0^{2\pi}\int_0^{2\pi} f_0(a, \nu t, \psi)\, e^{-i q \sigma \theta}\sin\psi\, d\nu t\, d\psi, \\ \frac{d\theta}{dt} &= \omega - \frac{p}{q}\nu - \frac{\varepsilon q}{4\pi^2 a\nu p}\sum_{\sigma} e^{i q \sigma \theta}\int_0^{2\pi}\int_0^{2\pi} f_0(a, \nu t, \psi)\, e^{-i q \sigma \theta}\cos\psi\, d\nu t\, d\psi. \end{aligned}\right\} \quad (13.25)$$

Knowing the expressions for $u_1(a, \nu t, \frac{p}{q}\nu t + \theta)$, $A_1(a, \theta)$ and $B_1(a, \theta)$, we can, in accordance with eq.(13.15), find an explicit expression for $f_1(a, \nu t + \theta)$, after which eq.(13.13) will yield the expressions for $A_2(a, \theta)$ and $B_2(a, \theta)$, which are necessary for constructing the second approximation:

$$\left.\begin{aligned} A_2(a, \theta) &= -\frac{q}{2\nu p}\left[a\frac{\partial B_1}{\partial a}A_1 + a\frac{\partial B_1}{\partial \theta}B_1 + 2A_1B_1\right] - \\ &- \frac{q}{4\pi^2\nu p}\sum_{\sigma} e^{i q \sigma \theta}\int_0^{2\pi}\int_0^{2\pi} f_1\left(a, \nu t, \frac{p}{q}\nu t + \theta\right) e^{-i q \sigma \theta}\sin\psi\, d\nu t\, d\psi, \\ B_2(a, \theta) &= \frac{q}{2a\nu p}\left[\frac{\partial A_1}{\partial a}A_1 + \frac{\partial A_1}{\partial \theta}B_1 - aB_1^2\right] - \end{aligned}\right\} \quad (13.26)$$

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$$-\frac{q}{4\pi^2 a\nu p}\sum_{\sigma} e^{iq\sigma\vartheta}\int_0^{2\pi}\int_0^{2\pi} f_1\left(a,\ \nu t,\ \frac{p}{q}\nu t+\vartheta\right)e^{-iq\sigma\vartheta}\cos\psi\, d\nu t\, d\psi.\Bigg]$$

Let us now consider the most general case.

Let it be required to investigate the behavior of a quasi-resonant system for the approaches to the resonant zone from a nonresonant zone. For this purpose, it is necessary to construct approximate solutions that permit studying the behavior of the system for a sufficiently large frequency interval and which, in special cases, will yield the above-presented formulas for the resonant as well as for the nonresonant case.

Here the detuning can no longer be considered small so that the approximate solution must be determined directly for eq.(12.1). Moreover, the dependence on the angle of phase shift must be introduced in the expressions for the instantaneous amplitude and frequency.

Thus, the solution, as done above, can be obtained in the form of the series

$$x = a\cos\left(\frac{p}{q}\nu t+\vartheta\right)+\varepsilon u_1(a,\ \nu t,\ \psi)+\varepsilon^2 u_2(a,\ \nu t,\ \psi)+\ldots, \qquad (13.27)$$

where a and θ must be determined from the following system of differential equations:

$$\left.\begin{aligned}\frac{da}{dt} &= \varepsilon A_1(a,\ \vartheta)+\varepsilon^2 A_2(a,\ \vartheta)+\ldots,\\ \frac{d\vartheta}{dt} &= \omega-\frac{p}{q}\nu+\varepsilon B_1(a,\ \vartheta)+\varepsilon^2 B_2(a,\ \vartheta)+\ldots,\end{aligned}\right\} \qquad (13.28)$$

and where, in this case, the difference $\omega - \frac{p}{q}\nu$ is not necessarily small.

Let us then determine the functions $u_i(a,\ \nu t,\ \psi)$, $A_i(a,\ \theta)$, $B_i(a,\ \theta)$, for i = 1, 2,... . For this purpose, we first use the system (13.28) to obtain

$$\left.\begin{aligned}\frac{d^2a}{dt^2} &= \varepsilon\left(\omega-\frac{p}{q}\nu\right)\frac{\partial A_1}{\partial\vartheta}+\varepsilon^2\left\{\frac{\partial A_1}{\partial a}A_1+\frac{\partial A_1}{\partial\vartheta}B_1+\left(\omega-\frac{p}{q}\nu\right)\frac{\partial A_2}{\partial\vartheta}\right\}+\varepsilon^3\ldots,\\ \frac{d^2\vartheta}{dt^2} &= \varepsilon\left(\omega-\frac{p}{q}\nu\right)\frac{\partial B_1}{\partial\vartheta}+\varepsilon^2\left\{\frac{\partial B_1}{\partial a}A_1+\frac{\partial B_1}{\partial\vartheta}B_1+\left(\omega-\frac{p}{q}\nu\right)\frac{\partial B_2}{\partial\vartheta}\right\}+\varepsilon^3\ldots,\\ \left(\frac{da}{dt}\right)^2 &= \varepsilon^2 A_1^2+\varepsilon^3\ldots,\\ \frac{da}{dt}\frac{d\vartheta}{dt} &= \varepsilon\left(\omega-\frac{p}{q}\nu\right)A_1+\varepsilon^2\left\{A_1B_1+\left(\omega-\frac{p}{q}\nu\right)A_2\right\}+\varepsilon^3\ldots,\end{aligned}\right\} \qquad (13.29)$$

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$$\left(\frac{d\vartheta}{dt}\right)^2 = \left(\omega - \frac{p}{q}\nu\right)^2 + \varepsilon 2\left(\omega - \frac{p}{q}\nu\right)B_1 + \\ + \varepsilon^2\left\{B_1^2 + 2B_2\left(\omega - \frac{p}{q}\nu\right)\right\} + \varepsilon^3 \ldots,$$

after which, on substituting the value of x from eq.(13.5) and that of $\frac{d^2x}{dt^2}$ from equation (13.8) in the right side of eq.(12.1) and bearing in mind eqs.(13.28), and (13.29), we obtain the following expression after arranging the results by powers of the small parameter:

$$\begin{aligned}
\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon\Bigg\{&\left[\left(\omega - \frac{p}{q}\nu\right)\frac{\partial A_1}{\partial\vartheta} - 2\omega a B_1\right]\cos\left(\frac{p}{q}\nu t + \vartheta\right) - \\
&- \left[\left(\omega - \frac{p}{q}\nu\right)a\frac{\partial B_1}{\partial\vartheta} + 2\omega A_1\right]\sin\left(\frac{p}{q}\nu t + \vartheta\right) + \\
&+ \frac{\partial^2 u_1}{\partial t^2} + 2\frac{\partial^2 u_1}{\partial\vartheta\,\partial t}\left(\omega - \frac{p}{q}\nu\right) + \frac{\partial^2 u_1}{\partial\vartheta^2}\left(\omega - \frac{p}{q}\nu\right)^2 + \omega^2 u_1\Bigg\} + \\
&+ \varepsilon^2\Bigg\{\left[\left(\omega - \frac{p}{q}\nu\right)\frac{\partial A_2}{\partial\vartheta} - 2\omega a B_2\right]\cos\left(\frac{p}{q}\nu t + \vartheta\right) - \\
&- \left[\left(\omega - \frac{p}{q}\nu\right)a\frac{\partial B_2}{\partial\vartheta} + 2\omega A_2\right]\sin\left(\frac{p}{q}\nu t + \vartheta\right) + \frac{\partial^2 u_2}{\partial t^2} + \\
&+ 2\frac{\partial^2 u_2}{\partial\vartheta\,\partial t}\left(\omega - \frac{p}{q}\nu\right) + \frac{\partial^2 u_2}{\partial\vartheta^2}\left(\omega - \frac{p}{q}\nu\right)^2 + \omega^2 u_2 + \\
&+ \left[\frac{\partial A_1}{\partial a}A_1 + \frac{\partial A_1}{\partial\vartheta}B_1 - B_1^2 a\right]\cos\left(\frac{p}{q}\nu t + \vartheta\right) - \\
&- \left[\frac{\partial B_1}{\partial a}A_1 + \frac{\partial B_1}{\partial\vartheta}B_1 + 2A_1B_1\right]\sin\left(\frac{p}{q}\nu t + \vartheta\right) + \\
&+ \frac{\partial u_1}{\partial a}\left(\omega - \frac{p}{q}\nu\right)\frac{\partial A_1}{\partial\vartheta} + \frac{\partial u_1}{\partial\vartheta}\left(\omega - \frac{p}{q}\nu\right)\frac{\partial B_1}{\partial\vartheta} + \\
&+ 2\frac{\partial^2 u_1}{\partial\vartheta^2}\left(\omega - \frac{p}{q}\nu\right)B_1 + 2\frac{\partial^2 u_1}{\partial a\,\partial t}A_1 + 2\frac{\partial^2 u_1}{\partial\vartheta\,\partial t}B_1 + \\
&+ 2\frac{\partial^2 u_1}{\partial a\,\partial\vartheta}\left(\omega - \frac{p}{q}\nu\right)A_1\Bigg\} + \varepsilon^3 \ldots
\end{aligned} \tag{13.30}$$

The right side of eq.(12.1), according to eqs.(13.7), (13.27), (13.28), and (13.29), can be represented in the form

$$\begin{aligned}
&\varepsilon f\left(\nu t,\ x,\ \frac{dx}{dt}\right) = \\
&= \varepsilon f\left(\nu t,\ a\cos\left(\frac{p}{q}\nu t + \vartheta\right),\ -a\omega\sin\left(\frac{p}{q}\nu t + \vartheta\right)\right) + \\
&+ \varepsilon^2\left[f'_x\left(\nu t,\ a\cos\left(\frac{p}{q}\nu t + \vartheta\right),\ -a\omega\sin\left(\frac{p}{q}\nu t + \vartheta\right)\right)u_1 + \right.
\end{aligned} \tag{13.31}$$

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$$+f'_{x'}\left(\nu t,\, a\cos\left(\frac{p}{q}\nu t+\vartheta\right),\, -a\omega\sin\left(\frac{p}{q}\nu t+\vartheta\right)\right)\left(A_1\cos\left(\frac{p}{q}\nu t+\vartheta\right)-\right.$$

$$\left.\left.-aB_1\sin\left(\frac{p}{q}\nu t+\vartheta\right)+\frac{\partial u_1}{\partial\vartheta}\left(\omega-\frac{p}{q}\nu\right)\right)\right]+\varepsilon^3\ldots \quad (13.31)$$

By equating the coefficients of the same powers of ε in the right sides of equations (13.30) and (13.31), we obtain the following system of differential equations:

$$\frac{\partial^2 u_1}{\partial t^2}+2\frac{\partial^2 u_1}{\partial\vartheta\,\partial t}\left(\omega-\frac{p}{q}\nu\right)+\frac{\partial^2 u_1}{\partial\vartheta^2}\left(\omega-\frac{p}{q}\nu\right)^2+\omega^2 u_1=$$

$$=f_0\left(a,\nu t,\frac{p}{q}\nu t+\vartheta\right)+\left[\left(\omega-\frac{p}{q}\nu\right)a\frac{\partial B_1}{\partial\vartheta}+2\omega A_1\right]\sin\left(\frac{p}{q}\nu t+\vartheta\right)-$$

$$-\left[\left(\omega-\frac{p}{q}\nu\right)\frac{\partial A_1}{\partial\vartheta}-2a\omega B_1\right]\cos\left(\frac{p}{q}\nu t+\vartheta\right), \quad (13.32)$$

$$\frac{\partial^2 u_2}{\partial t^2}+2\frac{\partial^2 u_2}{\partial\vartheta\,\partial t}\left(\omega-\frac{p}{q}\nu\right)+\frac{\partial^2 u_2}{\partial\vartheta^2}\left(\omega-\frac{p}{q}\nu\right)^2+\omega^2 u_2=$$

$$=f_1\left(a,\nu t,\frac{p}{q}\nu t+\vartheta\right)+\left\{\left(\omega-\frac{p}{q}\nu\right)a\frac{\partial B_2}{\partial\vartheta}+2\omega A_2+\left(a\frac{\partial B_1}{\partial a}A_1+\right.\right.$$

$$\left.\left.+a\frac{\partial B_1}{\partial\vartheta}B_1+2A_1B_1\right)\right\}\sin\left(\frac{p}{q}\nu t+\vartheta\right)-\left\{\left(\omega-\frac{p}{q}\nu\right)\frac{\partial A_2}{\partial\vartheta}-\right.$$

$$\left.-2\omega aB_2+\left(\frac{\partial A_1}{\partial a}A_1+\frac{\partial A_1}{\partial\vartheta}B_1-aB_1^2\right)\right\}\cos\left(\frac{p}{q}\nu t+\vartheta\right),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \quad (13.33)$$

where the following symbols have been introduced:

$$f_0\left(a,\nu t,\frac{p}{q}\nu t+\vartheta\right)=f\left(\nu t,\, a\cos\left(\frac{p}{q}\nu t+\vartheta\right),\, -a\omega\sin\left(\frac{p}{q}\nu t+\vartheta\right)\right),$$

$$f_1\left(a,\nu t,\frac{p}{q}\nu t+\vartheta\right)=f'_x\left(\nu t,\, a\cos\left(\frac{p}{q}\nu t+\vartheta\right),\, -a\omega\sin\left(\frac{p}{q}\nu t+\vartheta\right)\right)u_1+$$

$$+f'_{x'}\left(\nu t,\, a\cos\left(\frac{p}{q}\nu t+\vartheta\right),\, -a\omega\sin\left(\frac{p}{q}\nu t+\vartheta\right)\right)\times$$

$$\times\left(A_1\cos\left(\frac{p}{q}\nu t+\vartheta\right)-aB_1\sin\left(\frac{p}{q}\nu t+\vartheta\right)+\frac{\partial u_1}{\partial\vartheta}\left(\omega-\frac{p}{q}\nu\right)\right)- \quad (13.34$$

$$-\frac{\partial u_1}{\partial a}\left(\omega-\frac{p}{q}\nu\right)\frac{\partial A_1}{\partial\vartheta}-\frac{\partial u_1}{\partial\vartheta}\left(\omega-\frac{p}{q}\nu\right)\frac{\partial B_1}{\partial\vartheta}-2\frac{\partial^2 u_1}{\partial\vartheta^2}B_1\left(\omega-\frac{p}{q}\nu\right)-$$

$$-2\frac{\partial^2 u_1}{\partial a\,\partial t}A_1-2\frac{\partial^2 u_1}{\partial\vartheta\,\partial t}B_1-2\frac{\partial^2 u_1}{\partial a\,\partial\vartheta}\left(\omega-\frac{p}{q}\nu\right)A_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

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For determining $u_1(a, \nu t, \frac{p}{q} \nu t + \theta)$, $A_1(a, \theta)$ and $B_1(a, \theta)$ from eq. (13.32), we proceed as in the resonant case; the condition that the expressions for $u_1(a, \nu t, \frac{p}{q} \nu t + \theta)$ must contain no terms whose denominators might vanish (at $\omega = \frac{p}{q} \nu$), permits a unique selection of the corresponding expressions for $A_1(a, \theta)$ and $B_1(a, \theta)$.

After a series of calculations analogous to those given for the resonant case, we find, for $u_1(a, \nu t, \frac{p}{q} \nu t + \theta)$, the expression

$$u_1\left(a, \nu t, \frac{p}{q}\nu t + \theta\right) = \frac{1}{4\pi^2} \sum_{\substack{n \quad m \\ [nq+(m\pm 1)p \neq 0]}} \sum \frac{e^{i\left\{n\nu t + m\left(\frac{p}{q}\nu t + \theta\right)\right\}}}{\omega^2 - (n\nu + m\omega)^2} \times \tag{13.35}$$

$$\times \int_0^{2\pi} \int_0^{2\pi} f_0\left(a, \nu t, \frac{p}{q}\nu t + \theta\right) e^{-i\left\{n\nu t + m\left(\frac{p}{q}\nu t + \theta\right)\right\}} \, d\nu t \, d\psi,$$

and for determining $A_1(a, \theta)$ and $B_1(a, \theta)$ we obtain the following systems of equations:

$$\left.\begin{aligned} &\left(\omega - \frac{p}{q}\nu\right)\frac{\partial A_1}{\partial \theta} - 2a\omega B_1 = \\ &= \frac{1}{2\pi^2} \sum_{\sigma} e^{i\sigma q\theta} \int_0^{2\pi} \int_0^{2\pi} f_0\left(a, \nu t, \frac{p}{q}\nu t + \theta\right) e^{-iq\sigma\theta} \cos\psi \, d\nu t \, d\psi, \\ &\left(\omega - \frac{p}{q}\nu\right) a \frac{\partial B_1}{\partial \theta} + 2\omega A_1 = \\ &= -\frac{1}{2\pi^2} \sum_{\sigma} e^{i\sigma q\theta} \int_0^{2\pi} \int_0^{2\pi} f_0\left(a, \nu t, \frac{p}{q}\nu t + \theta\right) e^{-iq\sigma\theta} \sin\psi \, d\nu t \, d\psi. \end{aligned}\right\} \tag{13.36}$$

Since the expressions for $A_1(a, \theta)$ and $B_1(a, \theta)$ must be chosen so as to satisfy the condition that $u_1(a, \nu t, \frac{p}{q} \nu t + \theta)$ must be finite, we can select, as the wanted expressions, a certain partial solution, periodic in θ, for the system (13.36), which gives us no difficulty.

The explicit expressions for $A_1(a, \theta)$ and $B_1(a, \theta)$ will have the following form:

$$A_1(a, \theta) = \frac{1}{2\pi^2} \sum_{\sigma} e^{iq\sigma\theta} \left\{ \frac{(q\omega - p\nu)\sigma a \int_0^{2\pi} \int_0^{2\pi} f_0 e^{-iq\sigma\theta} \cos\psi \, d\nu t \, d\psi}{4\omega^2 - (q\omega - p\nu)^2 \sigma^2} - \right.$$

$$
\left.\begin{aligned}
&\qquad\qquad - \frac{2\omega \int\limits_0^{2\pi}\int\limits_0^{2\pi} f_0 e^{-iq\sigma\vartheta} \sin\psi \, d\nu t\, d\psi}{4\omega^2 - (q\omega - p\nu)^2\sigma^2}\Bigg\}, \\
&B_1(a,\vartheta) = -\frac{1}{2\pi^2 a}\sum_\sigma e^{iq\sigma\vartheta}\Bigg\{\frac{(q\omega - p\nu)\sigma \int\limits_0^{2\pi}\int\limits_0^{2\pi} f_0 e^{-iq\sigma\vartheta}\sin\psi\, d\nu t\, d\psi}{4\omega^2 - (q\omega - p\nu)^2\sigma^2} + \\
&\qquad\qquad + \frac{2\omega \int\limits_0^{2\pi}\int\limits_0^{2\pi} f_0 e^{-iq\sigma\vartheta}\cos\psi\, d\nu t\, d\psi}{4\omega^2 - (q\omega - p\nu)^2\sigma^2}\Bigg\}.
\end{aligned}\right\} \qquad (13.37)
$$

After having found $u_1(a,\ \nu t,\ \frac{p}{q}\nu t + \theta)$, $A_1(a,\ \theta)$, $B_1(a,\ \theta)$, eq.(13.33), can be used for determining all quantities necessary for the construction of the solution in second approximation.

Thus, for determining $A_2(a,\ \theta)$ and $B_2(a,\ \theta)$ we obtain the system

$$
\left.\begin{aligned}
&\left(\omega - \frac{p}{q}\nu\right)\frac{\partial A_2}{\partial\vartheta} - 2a\omega B_2 = -\left\{\frac{\partial A_1}{\partial a}A_1 + \frac{\partial A_1}{\partial\vartheta}B_1 - aB_1^2\right\} + \\
&+ \frac{1}{2\pi^2}\sum_\sigma e^{iq\sigma\vartheta}\int\limits_0^{2\pi}\int\limits_0^{2\pi} f_1\left(a,\ \nu t,\ \frac{p}{q}\nu t + \vartheta\right)e^{-iq\sigma\vartheta}\cos\psi\, d\nu t\, d\psi, \\
&\left(\omega - \frac{p}{q}\nu\right)a\frac{\partial B_2}{\partial\vartheta} + 2\omega A_2 = -\left\{a\frac{\partial B_1}{\partial a}A_1 + a\frac{\partial B_1}{\partial\vartheta}B_1 + 2A_1B_1\right\} - \\
&- \frac{1}{2\pi^2}\sum_\sigma e^{iq\sigma\vartheta}\int\limits_0^{2\pi}\int\limits_0^{2\pi} f_1\left(a,\ \nu t,\ \frac{p}{q}\nu t + \vartheta\right)e^{-iq\sigma\vartheta}\sin\psi\, d\nu t\, d\psi.
\end{aligned}\right\} \qquad (13.38)
$$

It is obvious that the formulas derived by us for studying the resonant region as well as the approaches to it, will yield all formulas found earlier. Thus, by putting $\omega - \frac{p}{q}\nu = \varepsilon\Delta$ in eq.(13.36), we find with an accuracy to terms of the first order of smallness, expressions for $A_1(a,\ \theta)$ and $B_1(a,\ \theta)$ of eq.(13.23), which were obtained in the case of resonance.

Assuming that the oscillatory system is out of resonance, we have $\omega t - \frac{p}{q}\nu t \neq$ const, so that eq.(13.36) will yield expressions for $A_1(a,\ \theta)$ and $B_1(a,\ \theta)$ coinciding with those of eq.(12.35) obtained for the nonresonant case.

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To summarize, we present a scheme for the construction of a solution of eq.(12.1) in first and second approximations, for the most general case. As the first approximation we take

$$x = a\cos\left(\frac{p}{q}\nu t + \theta\right), \tag{13.39}$$

where a and θ must be determined from the equations

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a, \theta), \\ \frac{d\theta}{dt} &= \omega - \frac{p}{q}\nu + \varepsilon B_1(a, \theta), \end{aligned}\right\} \tag{13.40}$$

in which $A_1(a, \theta)$ and $B_1(a, \theta)$ are partial, periodic solutions of the system (13.36). In second approximation we put

$$x = a\cos\left(\frac{p}{q}\nu t + \theta\right) + \varepsilon u_1\left(a, \nu t, \frac{p}{q}\nu t + \theta\right), \tag{13.41}$$

where a and θ are determined by the equations:

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a, \theta) + \varepsilon^2 A_2(a, \theta), \\ \frac{d\theta}{dt} &= \omega - \frac{p}{q}\nu + \varepsilon B_1(a, \theta) + \varepsilon^2 B_2(a, \theta), \end{aligned}\right\} \tag{13.42}$$

in which $A_1(a, \theta)$, $B_1(a, \theta)$, $A_2(a, \theta)$, $B_2(a, \theta)$ must be found from the systems (13.36) and (13.38) and $u_1(a, \nu t, \frac{p}{q}\nu t + \theta)$ from eq.(13.35).

We note again that the equation of second approximation (13.42) taking account of the expressions for $A_2(a, \theta)$ and $B_2(a, \theta)$ of eq.(13.38) appear to be rather complicated only because they are written in the most general form. For concrete examples, even in second approximation, we obtain relatively simple equations determining the amplitude and phase of the oscillation [cf., for example, eqs.(12.56) and (14.38)].

Let us consider the first approximation.

In contrast to the nonresonant case, here the variables are not separated in the equations of first approximation (13.40), and we have a system of two mutually connected equations for determining the two unknowns a and θ.

We note first that, for sufficiently large values of p and q, in view of the assumption made earlier that the functions $f_n(x, \frac{dx}{dt})$ are of polynomial character,

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the first approximation in the resonant case will not differ from the nonresonant case. In fact, for sufficiently large values of p and q only terms corresponding to $\sigma = 0$ will remain in the sums on the right sides of eq.(13.40), which terms will coincide with the expressions (12.35) obtained in the nonresonant case.

Thus the effect of resonance will be manifested, generally speaking, at small values of the numbers p and q.

Let us return to a consideration of the equations of first approximation (13.40).

Since the right sides of these equations depend on a and θ, we are unable, in the general case, to integrate them in a closed form. The qualitative character of the solutions can be investigated in the general case, by means of the Poincaré theory, since here we have to do with two first order equations.

According to the fundamental results of this theory (cf. Chapter II) we may assert that any solution* of eq.(13.40), with increasing time, approaches either the constant solution

$$a = a_i, \quad \vartheta = \vartheta_i \quad (i = 1, 2, \ldots),$$

determined from the equation

$$A_1(a, \vartheta) = 0, \quad \omega - \frac{p}{q}\nu + \varepsilon B_1(a, \vartheta) = 0, \tag{13.43}$$

or a periodic solution.

Thus, we obtain two basic types of stationary oscillations: oscillations corresponding to a constant solution or, as they say, to the "equilibrium point" of eq.(13.40), and those corresponding to a periodic solution.

In the former case, the oscillations (in first approximation) take place at a frequency exactly equal to $\frac{p}{q}\nu$ which is, consequently, at a simple ratio to the excitation frequency. For this reason such a state of oscillations is called synchronous.

In the higher approximations [cf. e.g., eq.(13.21)] the expression for

* It must be borne in mind that, for every solution, the quantity a remains finite. From the physical point of view this limitation is always satisfied, since the oscillation amplitude cannot increase without limit.

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$u_1(a, \nu t, \frac{p}{q}\nu t + \theta)$, generally speaking, contains in addition to the fundamental frequency $\frac{p\nu}{q}$ other overtones of the subfrequency $\frac{\nu}{q}$.

In the case where a constant solution of the type $a = 0$ exists in the system, corresponding to the absence of natural oscillations, the expression for $u_1(a, \nu t, \frac{p}{q}\nu t + \theta)$ of eq.(13.21) will be the same as in the nonresonant case of eq.(12.34), and will represent a heteroperiodic state of oscillation.

Let us study the question of the stability of the stationary synchronous state. To determine the stability of the constant solutions a_0 and θ_0 determined by equation (13.43), the corresponding equations of variation must be set up.

On the basis of eq.(13.40), the equations of variation can be written in the form

$$\left.\begin{aligned} \frac{d\delta a}{dt} &= \varepsilon A'_{1a}(a_0, \vartheta_0)\delta a + \varepsilon A'_{1\vartheta}(a_0, \vartheta_0)\delta\vartheta, \\ \frac{d\delta\vartheta}{dt} &= \varepsilon B'_{1a}(a_0, \vartheta_0)\delta a + \varepsilon B'_{1\vartheta}(a_0, \vartheta_0)\delta\vartheta. \end{aligned}\right\} \qquad (13.44)$$

The characteristic equation for the system (13.44) will then be

$$\begin{vmatrix} \varepsilon A'_{1a} - \lambda & \varepsilon A'_{1\vartheta} \\ \varepsilon B'_{1a} & \varepsilon B'_{1\vartheta} - \lambda \end{vmatrix} = 0$$

or

$$\lambda^2 - (\varepsilon A'_{1a} + \varepsilon B'_{1\vartheta})\lambda + \varepsilon^2(A'_{1a}B'_{1\vartheta} - B'_{1a}A'_{1\vartheta}) = 0. \qquad (13.45)$$

From eq.(13.45) we obtain the following conditions for stability of the synchronous stationary oscillations under consideration:

$$A'_{1a}(a_0, \vartheta_0) + B'_{1\vartheta}(a_0, \vartheta_0) < 0, \qquad (13.46)$$

$$A'_{1a}(a_0, \vartheta_0)B'_{1\vartheta}(a_0, \vartheta_0) - A'_{1\vartheta}(a_0, \vartheta_0)B'_{1a}(a_0, \vartheta_0) > 0. \qquad (13.47)$$

In the latter case, corresponding to the periodic solution of eq.(13.40), the oscillation will occur in first approximation with two fundamental frequencies, i.e., the frequency ω or $\frac{p}{q}\nu + \Delta\omega$ and the beat frequency $\Delta\omega$, where $\Delta\omega = \frac{2\pi}{T}$ (T = period of the given periodic solution). These oscillations are called asynchronous.

As an example of illustrating the character of the synchronous and asynchronous states, let us consider the vacuum-tube oscillator under the influence of an ex-

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ternal periodic force of frequency ν.

In the case where the oscillator is set up in accordance with the diagram given in Fig.78, the differential equation describing the oscillatory process will be

$$\frac{d^2e}{dt^2}+\omega^2 e=-\omega^2\left[\frac{L}{R}\frac{de}{dt}-(M-DL)\frac{di_a}{dt}\right], \quad (13.48)$$

where L is the inductance; M the coefficient of mutual inductance; R the resistance; C the capacitance; D the transconductance of the tube; $\omega = \frac{1}{\sqrt{LC}}$ the natural frequency of the line circuit; $i_a = f(E_o + F\cos\nu t + e)$ the tube characteristic (i_a = plate current); $E = E_o + F\cos\nu t + e$ the total control voltage; E_o the constant component of the total control voltage; $F\cos\nu t$ the component of the nonlinear control voltage due to external excitation; e the component of the control voltage due to oscillations in the circuit.

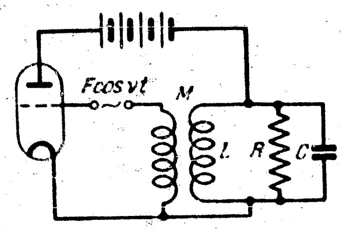

Fig.78

Let us consider the case when $f(E_o + u)$, where $u = e + F\cos\nu t$, is a cubic polynomial

$$f(E_0+u)=f(E_0)+S_0 u+S_1 u^2-S_2 u^3, \quad (13.49)$$

in which $S_2 > 0$.

Assume that the terms on the right side of eq.(13.48) are small; then the oscillations will be close to harmonic, and we will be able to construct the approximate solution of eq.(13.48) by making use of the formulas presented above.

Thus, in the case where p = 1, q = 2, i.e., $\omega \approx \frac{\nu}{2}$, we have, in first approximation,

$$e = a\cos\left(\frac{\nu}{2}t+\theta\right), \quad (13.50)$$

where a and θ must be determined from the equations

$$\frac{da}{dt}=\delta_0\left[-\frac{3S_2}{4S_{cr}}a^3+\frac{S_0-S_{cr}-\frac{3}{2}S_2F^2}{S_{cr}}a\right]+\frac{aS_1F\delta_0}{2S_{cr}}\cos 2\theta, \quad (13.51)$$

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$$\frac{d\vartheta}{dt} = \omega - \frac{\nu}{2} - \frac{S_1 F \delta_0}{2S_{cr}} \sin 2\vartheta,$$

where $\delta_e = \frac{1}{2RC}$, $S_{cr} = \frac{L}{R(M - DL)}$.

In view of the fact that there is only one unknown function, namely θ in the second equation of the system (13.51), it may be integrated by quadrature.

Let us discuss still another question.

Let us find the relations between the frequencies ω and ν, and likewise between the coefficients of the polynomial (13.49) in the oscillator, at which stationary oscillations will exist.

Assume first that

$$\left|\omega - \frac{\nu}{2}\right| < \left|\frac{\delta_0 S_1 F}{2S_{cr}}\right|. \qquad (13.52)$$

Then, by integrating the second equation of the system (13.51), we get

$$\vartheta(t) \underset{t\to\infty}{\to} \vartheta_0, \qquad (13.53)$$

where

$$\vartheta_0 = \frac{1}{2} \arcsin 2 \frac{\omega - \frac{\nu}{2}}{\delta_0 S_1 F} S_{cr}. \qquad (13.54)$$

From the first equation of the system (13.51), when eq.(13.53) is satisfied, we find

$$a(t) \underset{t\to\infty}{\to} a_0, \qquad (13.55)$$

where a_e is determined from the equation:

$$\left(S_0 - S_{cr} - \frac{3}{2} S_2 F^2 + \frac{1}{2} S_1 F \cos 2\vartheta_0\right) a_0 - \frac{3}{4} S_2 a_0^3 = 0. \qquad (13.56)$$

If

$$S_0 - S_{cr} - \frac{3}{2} S_2 F^2 + \frac{1}{2} S_1 F \cos 2\vartheta_0 < 0, \qquad (13.57)$$

then, obviously

$$a_0 = 0.$$

Thus, when the conditions (13.52) and (13.57) are satisfied, a heteroperiodic

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state will be established in the oscillator, and only the stationary state is possible in this case.

Let now

$$S_0 - S_{cr} - \frac{3}{2} S_3 F^2 + \frac{1}{2} S_1 F \cos 2\vartheta_0 > 0; \qquad (13.58)$$

Then, the solution $a_0 = 0$ is unstable, and the system will be self-exciting.

From eq. (13.56) we find the value of a_0 as follows:

$$a_0 = \sqrt{\frac{4}{3S_0}\left[S_0 - S_{cr} - \frac{3}{2} S_3 F^2 + \frac{1}{2} S_1 F \cos 2\vartheta_0\right]}. \qquad (13.59)$$

Thus, when the conditions (13.52) and (13.58) are satisfied, a synchronous state is established in the oscillatory system under consideration, and in first approximation

$$e = a_0 \cos\left(\frac{\nu}{2} t + \vartheta_0\right), \qquad (13.60)$$

where a_0 and θ_0 are determined according to eq. (13.54) and (13.59).

Stationary oscillations with constant amplitude and phase and with a frequency equal to half the excitation frequency, are established in the oscillator.

According to condition (13.52), the detuning of the resonance $\left|\omega - \frac{\nu}{2}\right|$ must not exceed a certain value in this case. In other words, a synchronous state is possible at sufficiently small values for the detuning.

Let us now consider the case when the system is far from resonance, so that the condition

$$\left|\omega - \frac{\nu}{2}\right| > \left|\frac{\delta_0 S_1 F}{2S_{cr}}\right|. \qquad (13.61)$$

is satisfied.

Then, on integrating the second equation of system (13.51), θ may be represented in the form

$$\vartheta = \Delta\omega t + \Phi(\Delta\omega t + \theta), \qquad (13.62)$$

where θ is an arbitrary constant, $\Phi(\theta)$ is a periodic function of θ with the period 2π

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$$\Delta\omega = \frac{2\pi}{T}, \quad T = \int_0^{2\pi} \frac{d\vartheta}{\omega - \frac{\nu}{2} - \left|\frac{S_1 F \delta_0}{2S_{cr}}\right| \sin 2\vartheta}. \tag{13.63}$$

On substituting the value of θ from eq.(13.62) in the first equation of the system (13.51), we obtain a first-order equation with a periodic coefficient for the determination of the stationary amplitude values.

This equation will allow the solution a = 0, corresponding to the heteroperiodic state. The question of the stability of the solution a = 0 depends on the sign of the expression:

$$S_0 - S_{cr} - \frac{3}{2} S_2 F^2 + \frac{1}{2} S_1 F \overline{\cos 2\vartheta}, \tag{13.64}$$

where $\overline{\cos 2\theta}$ denotes the averaged value of cos 2θ over the period T:

$$\overline{\cos 2\vartheta} = \frac{1}{T} \int_0^T \cos 2\vartheta \, dt. \tag{13.65}$$

By virtue of eq.(13.63) we have $\overline{\cos 2\theta} = 0$, so that eq.(13.64) takes the form

$$S_0 - S_{cr} - \frac{3}{2} S_2 F^2. \tag{13.66}$$

If the expression (13.66) is negative, there is no self-excitation and the heteroperiodic state a = 0 is stable; but if the expression (13.66) is positive, self-excitation occurs in the system and, consequently, the heteroperiodic regime is unstable.

Thus, when the condition (13.61) and the condition

$$S_0 - S_{cr} - \frac{3}{2} S_2 F^2 > 0 \tag{13.67}$$

are satisfied, it may be shown that

$$a(t) \underset{t \to \infty}{\rightarrow} A(\Delta\omega t + \theta), \tag{13.68}$$

where $A(\Delta\omega t+\theta)$ is the corresponding periodic solution with period $T = \frac{2\pi}{\Delta\omega}$ of the first equation of the system (13.51) after substitution of the value of θ from equation (13.62) in it.

In particular, for sufficiently great values of the detuning $|\omega - \frac{\nu}{2}|$, we obtain approximately

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$$A(\theta) \approx a_0, \tag{13.69}$$

where

$$a^2 = \frac{4}{3S_0}\left(S_0 - S_{cr} - \frac{3}{2} S_2 F^2\right),$$

i.e., the value of the stationary amplitude in the nonresonant case.

Thus, when the conditions (13.69) and (13.67) are satisfied, a stationary two-frequency state - asynchronous oscillations - is established in the oscillator. In first approximation, for the stationary oscillations, we obtain the expression

$$e = A(\Delta\omega t + \theta)\cos\left[\left(\frac{\nu}{2} + \Delta\omega\right)t + \theta + \Phi(\Delta\omega t + \theta)\right], \tag{13.70}$$

in which $\frac{\nu}{2} + \Delta\omega$ is the fundamental frequency, while the amplitude $A(\Delta\omega t + \theta)$ and the phase $\theta + \Phi(\Delta\omega t + \theta)$ oscillate with the beat frequency $\Delta\omega$.

On analyzing expression (13.70), it is not difficult to note that, on moving away from resonance, the oscillations (13.70) approach the nonresonant oscillations of the form

$$e = a\cos(\omega t + \theta).$$

This obviously occurs in the most general case as well. On studying the general equations of first approximation, eq.(13.40), we may show that when the detuning $|\omega - \frac{\nu}{2}|$ increases, the "resonant first approximation" is continuously transformed into the "nonresonant" type.

Section 14. Influence of a Sinusoidal Force on a Nonlinear Oscillator

As a special case of the oscillatory system described by eq.(12.1), consider the nonlinear oscillator under the action of a harmonic force. The oscillations of such a system, as stated above, are described by the following differential equation

$$m\frac{d^2x}{dt^2} + kx = \varepsilon f\left(x, \frac{dx}{dt}\right) + \varepsilon E\sin\nu t\ ^{*)}. \tag{14.1}$$

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* (For footnote, see next page).

On analyzing this equation in our Introduction, we came to the conclusion that, in first approximation, only the fundamental resonance can be detected.

Thus, using the formulas already derived, let us construct approximate solutions of eq.(14.1) for the case of fundamental resonance (p = 1, q = 1).

Equations (13.39) and (13.40), in first approximation, yield

$$x = a\cos(\nu t + \theta), \tag{14.2}$$

where a and θ must be determined from the system of equations

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon}{2\pi\omega m}\int_0^{2\pi} f_0(a, \nu t + \theta)\sin(\nu t + \theta)\, d(\nu t + \theta) - \\ &\quad - \frac{\varepsilon E}{m(\omega + \nu)}\cos\theta, \\ \frac{d\theta}{dt} &= \omega - \nu - \frac{\varepsilon}{2\pi\omega a m}\int_0^{2\pi} f_0(a, \nu t + \theta)\cos(\nu t + \theta)\, d(\nu t + \theta) + \\ &\quad + \frac{\varepsilon E}{ma(\omega + \nu)}\sin\theta. \end{aligned}\right\} \tag{14.3}$$

Here

$$f_0(a, \nu t + \theta) = f(a\cos(\nu t + \theta), \; -a\omega\sin(\nu t + \theta)).$$

In second approximation we obtain

$$\begin{aligned} x &= a\cos(\nu t + \theta) + \\ &\quad + \frac{\varepsilon}{\pi}\sum_{m=2}^{N}\frac{1}{\omega^2(1 - m^2)}\Big[\cos m(\nu t + \theta)\times \\ &\quad \times \int_0^{2\pi} f_0(a, \nu t + \theta)\cos m(\nu t + \theta)\, d(\nu t + \theta) + \end{aligned} \tag{14.4}$$

* It is assumed in this case that the amplitude of the external harmonic force is small. If this conclusion is impossible because of physical considerations, we have the equation

$$m\frac{d^2x}{dt^2} + kx = \varepsilon f\left(x, \frac{dx}{dt}\right) + E\sin\nu t,$$

which, by the substitution $x = y + \dfrac{E}{k - \nu^2 m}$, is reduced to an equation of the type of eq.(12.1).

$$+ \sin m(\nu t+\theta)\int_0^{2\pi} f_0(a,\nu t+\theta)\sin m(\nu t+\theta)\,d(\nu t+\theta)\Big],$$

where a and θ are functions of time, and are determined by equations of the second approximation, which we will not write out.

Let us discuss in more detail the investigation of the first approximation.

As in the case of a nonlinear system subjected to a disturbance not explicitly depending on time, we put, for abbreviation [cf. eq.(7.4)],

$$\left.\begin{aligned} \lambda_e(a) &= \frac{\varepsilon}{\pi a\omega}\int_0^{2\pi} f_0(a,\nu t+\theta)\sin(\nu t+\theta)\,d(\nu t+\theta),\\ k_e(a) &= k-\frac{\varepsilon}{\pi a}\int_0^{2\pi} f_0(a,\nu t+\theta)\cos(\nu t+\theta)\,d(\nu t+\theta), \end{aligned}\right\} \quad (14.5)$$

and note that the parameters $\lambda_e(a)$, $k_e(a)$, so introduced are the equivalent coefficient of damping and the total equivalent coefficient of elasticity for the oscillatory system under consideration in the "free" state, in the absence of external excitation, i.e., for a system described by an equation of the form

$$m\frac{d^2x}{dt^2}+kx=\varepsilon f\left(x,\frac{dx}{dt}\right). \quad (14.6)$$

After this, eq.(14.3) may be written as follows:

$$\left.\begin{aligned} \frac{da}{dt} &= -\delta_e(a)\,a-\frac{\varepsilon E}{m(\omega+\nu)}\cos\theta,\\ \frac{d\theta}{dt} &= \omega_e(a)-\nu+\frac{\varepsilon E}{ma(\omega+\nu)}\sin\theta, \end{aligned}\right\} \quad (14.7)$$

where $\delta_e(a)=\frac{\lambda_e(a)}{2m}$, $\omega_e(a)=\sqrt{\frac{k_e(a)}{m}}$ are the equivalent decrement of damping and the equivalent frequency of the nonlinear natural oscillations described by equation (14.6).

Consider the stationary states of the oscillations. To obtain the stationary values of the amplitude a and the phase θ, in first approximation, the right sides of eq.(14.7) must be equated to zero, after which we obtain the relations

$$-\delta_e(a)a-\frac{\varepsilon E}{m(\omega+\nu)}\cos\theta=0, \quad (14.8)$$

$$\omega_e(a) - \nu + \frac{\varepsilon E}{ma(\omega + \nu)} \sin \theta = 0 \Bigg\}$$

or, with an accuracy to terms of the second order of smallness, the following relations:

$$\left. \begin{aligned} 2m\nu a\delta_e(a) &= -\varepsilon E \cos \theta, \\ ma\left[\omega_e^2(a) - \nu^2\right] &= -\varepsilon E \sin \theta, \end{aligned} \right\} \tag{14.9}$$

whence, on eliminating the phase θ, we find the relation between the amplitude of the stationary oscillations and the frequency of the external force:

$$m^2a^2\left[(\omega_e^2(a) - \nu^2)^2 + 4\nu^2\delta_e^2(a)\right] = \varepsilon^2E^2. \tag{14.10}$$

The resultant equations (14.9) and (14.10) coincide with the equations used in the classical linear theory for determining the amplitude and phase of forced oscillations

$$x = a \cos(\nu t + \theta) \tag{14.11}$$

in a system with the mass m, the coefficient of elasticity $k_e(a)$, and the coefficient of damping $\lambda_e(a)$ (and, respectively, with the frequency $\omega_e(a) = \sqrt{\frac{k_e(a)}{m}}$ and the decrement $\delta_e(a) = \frac{\lambda_e(a)}{2m}$), under the influence of the external sinusoidal force $\varepsilon E \sin \nu t$.

We may therefore formulate the following rule: Given a certain nonlinear system under the influence of an external sinusoidal force with a frequency close to the natural frequency of the system. Required, to find the values of the amplitude and phase of the stationary synchronous oscillations of eq.(14.2).

For this purpose, by linearizing the given oscillatory system in the free state (i.e., not taking into account the external force $\varepsilon E \sin \nu t$), we determine the functions of the amplitude: the equivalent decrement and the equivalent frequency of the natural oscillations.

On substituting the resultant value in the classical relations of the linear theory of oscillations (14.9) and (14.10), we obtain an equation for determining the required quantities, amplitude and phase.

This rule has been formulated for the special case of the oscillatory system

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described by the differential equation (14.1), but it may be extended to more general cases of oscillatory systems.

Let us introduce the conditions of stability for the synchronous stationary oscillations under consideration.

For the resonant case, the equations of first approximation (14.7) may be represented, with an accuracy to terms of the second order of smallness, in the form

$$\begin{aligned} 2\nu \frac{da}{dt} &= -2\nu a\delta_e(a) - \frac{\varepsilon E}{m}\cos\vartheta, \\ 2\nu a \frac{d\vartheta}{dt} &= [\omega_e^2(a) - \nu^2]\, a + \frac{\varepsilon E}{m}\sin\vartheta, \end{aligned} \tag{14.12}$$

and the equations of the stationary synchronous states, in the form

$$\left.\begin{aligned} R(a,\vartheta) &= 0, \\ \Phi(a,\vartheta) &= 0, \end{aligned}\right\} \tag{14.13}$$

where $R(a,\theta)$ and $\Phi(a,\theta)$ denote, respectively, the right sides of eq.(14.12).

Let a and θ be any solutions of the eq.(14.13). To investigate the question of their stability, let us make use of the conditions previously derived [cf. equation (13.46), (13.47)]. As applied to our case, they will have the following form

$$aR'_a(a,\vartheta) + \Phi'_\vartheta(a,\vartheta) < 0, \tag{14.14}$$

$$R'_a(a,\vartheta)\,\Phi'_\vartheta(a,\vartheta) - R'_\vartheta(a,\vartheta)\,\Phi'_a(a,\vartheta) > 0. \tag{14.15}$$

Let us discover the meaning of these inequalities.

From eq.(14.14) we have

$$aR'_a(a,\vartheta) + \Phi'_\vartheta(a,\vartheta) = -2\nu a\delta_e(a) - 2\nu a^2\frac{d\delta_e(a)}{da} + \frac{\varepsilon E}{m}\cos\vartheta,$$

whence, bearing in mind the first equation of the system (14.12), we find

$$\begin{aligned} aR'_a(a,\vartheta) &+ \Phi'_\vartheta(a,\vartheta) = \\ &= -2\nu a\frac{d\,(a\delta_e(a))}{da} - 2\nu a\delta_e(a) = -2\nu\frac{d\,(a^2\delta_e(a))}{da}. \end{aligned} \tag{14.16}$$

In view of the notation introduced in eq.(14.5), we may write

$$2\nu a^2\delta_e(a) = \frac{a^2\lambda_e(a)}{m}\,\nu = \frac{2\nu}{m\omega^2}\,W(a), \tag{14.17}$$

where $W(a) =$

$$\frac{1}{2\pi}\int_0^{2\pi} \varepsilon f(a\cos(\omega t+\vartheta), -a\omega\sin(\omega t+\vartheta))\, a\omega\sin(\omega t+\vartheta)\, d(\omega t+\vartheta) \quad (14.18)$$

represents the mean power dissipated by the force $\varepsilon f(x, \frac{dx}{dt})$ under the oscillations

$$x = a\cos(\omega t+\vartheta).$$

Under the ordinary law of friction, $W(a)$ increases with the amplitude, so that

$$W'(a) > 0.$$

In this way, if we confine ourselves to the consideration of systems obeying the ordinary law of friction, then condition (14.14), according to eqs.(14.16) and (14.17), will always be satisfied.

Consider now the condition (14.15). For this purpose, let us investigate the dependence of a and θ, the solutions of eq.(14.13), on the frequency ν.

On differentiating eq.(14.13) with respect to ν, we obtain

$$\left.\begin{aligned} R'_a\frac{da}{d\nu} + R'_\vartheta\frac{d\vartheta}{d\nu} &= -R'_\nu, \\ \Phi'_a\frac{da}{d\nu} + \Phi'_\vartheta\frac{d\vartheta}{d\nu} &= -\Phi'_\nu, \end{aligned}\right\} \quad (14.19)$$

whence we find

$$(R'_a\Phi'_\vartheta - \Phi'_a R'_\vartheta)\frac{da}{d\nu} = \Phi'_\nu R'_\vartheta - R'_\nu\Phi'_\vartheta. \quad (14.20)$$

On the other hand, eq.(14.13) yields

$$\left.\begin{aligned} R'_\vartheta &= \frac{\varepsilon E}{m}\sin\vartheta, \quad R'_\nu = -2\delta_e(a)\,a, \\ \Phi'_\vartheta &= \frac{\varepsilon E}{m}\cos\vartheta, \quad \Phi'_\nu = -2\nu a, \end{aligned}\right\} \quad (14.21)$$

in connection with which the right side of eq.(14.20) may be written as follows:

$$\Phi'_\nu R'_\vartheta - R'_\nu\Phi'_\vartheta = 2a\left(-\nu\frac{E}{m}\sin\vartheta + \delta_e(a)\frac{E}{m}\cos\vartheta\right),$$

or, bearing in mind eq.(14.9), in the form

$$\Phi'_\nu R'_\vartheta - R'_\nu\Phi'_\vartheta = 2\nu a^2\left[(\omega_e^2(a) - \nu^2) - 2\delta_e^2(a)\right]. \quad (14.22)$$

It thus follows from eqs.(14.20) and (14.22) that

$$(R'_a\Phi'_\vartheta - \Phi'_a R'_\vartheta)\frac{da}{d\nu} = 2\nu a^2\left[(\omega_e^2(a) - \nu^2) - 2\delta_e^2(a)\right].$$

It is now obvious that the condition of stability (14.15) may be represented in the form

$$\left.\begin{aligned} \frac{da}{d\nu} > 0, \quad & if \quad \omega_e^2(a) > \nu^2 + 2\delta_e^2(a), \\ \frac{da}{d\nu} < 0, \quad & if \quad \omega_e^2(a) < \nu^2 + 2\delta_e^2(a), \end{aligned}\right\} \tag{14.23}$$

or, with an accuracy to terms of the first order of smallness [$\delta_e(a)$ being a term of the second order of smallness],

$$\left.\begin{aligned} \frac{da}{d\nu} > 0, \quad & if \quad \omega_e(a) > \nu, \\ \frac{da}{d\nu} < 0, \quad & if \quad \omega_e(a) < \nu. \end{aligned}\right\} \tag{14.24}$$

The conditions of stability (14.24) so obtained are very convenient for the graphic representation of the relation between amplitude and frequency.

Now, making use of eq. (14.10), let us construct the curve (Fig.79) of

$$a = F(\nu) \tag{14.25}$$

(the resonance curve), and also the curve

$$a = F_0(\nu), \tag{14.26}$$

defined by the equation of exact resonance

$$\omega_e(a) = \nu$$

(the so-called skeleton curve).

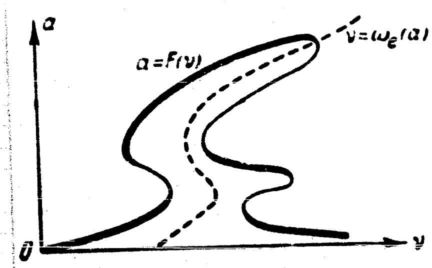

Fig.79

Then on the branch of the curve (14.25) lying to the left of curve (14.26) those parts are stable (i.e., correspond to stable amplitudes) on which a increases with ν. Conversely, on the branch lying to the right of curve (14.26), those segments will be stable on which a decreases with increasing ν.

The graphical construction makes the relation between the stable stationary amplitude and the frequency of the exciting force clear and, in particular, permits determining the points of discontinuity due to the phenomena of hysteresis, which are characteristic only for nonlinear systems.

As a concrete example, consider the nonlinear oscillator with a nonlinear restoring force having a hard characteristic ($F = cx + dx^3$), under the influence of an external sinusoidal force. Let the oscillations of the vibrator be described by an equation of the form

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx + dx^3 = E \sin \nu t, \qquad (14.27)$$

where x is a coordinate defining the position of the system; t the time; m the mass of the system; b the coefficient of resistance; $F = cx + dx^3$ the nonlinear elastic restoring force; E and ν, respectively, the amplitude and frequency of the external sinusoidal force.

Let us introduce, for simplification the dimensionless x_1 and t_1 of the formulas

$$x_1 = \sqrt{\frac{c}{d}}\, x, \quad t_1 = \sqrt{\frac{c}{m}}\, t.$$

Equation (14.27) may then be presented in the form

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + x + x^3 = E_1 \sin \nu t, \qquad (14.28)$$

where $\delta = \frac{b}{mc}$; $E_1 = \frac{E}{c}\sqrt{\frac{d}{c}}$, and for simplification, the subscripts of x and t have been omitted.

We assume now that, in the system under investigation, the friction as well as the amplitude of the external force are small and, in addition, that the characteristic of the nonlinear restoring force is sufficiently close to linear.

Then, a comparison of eq.(14.28) with eq.(14.1) will give

$$\varepsilon f\left(x, \frac{dx}{dt}\right) = -\delta\frac{dx}{dt} - x^3, \ \varepsilon E = E_1, \qquad (14.29)$$

after which, making use of eqs.(14.2), (14.5), and (14.7), we obtain in first approximation the solution of eq.(14.28) for the case of the fundamental resonance, in the form

$$x = a \cos(\nu t + \theta), \qquad (14.30)$$

where a and θ must be determined from the system of equations

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$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\delta a}{2} - \frac{E_1}{1+\nu}\cos\theta, \\ \frac{d\theta}{dt} &= 1-\nu+\frac{3a^2}{8}+\frac{E_1}{a(1+\nu)}\sin\theta. \end{aligned}\right\} \qquad (14.31)$$

Let us now pass directly to a consideration of the stationary state of synchronous oscillations. In such a state and according to eq.(14.30), the quantity x,

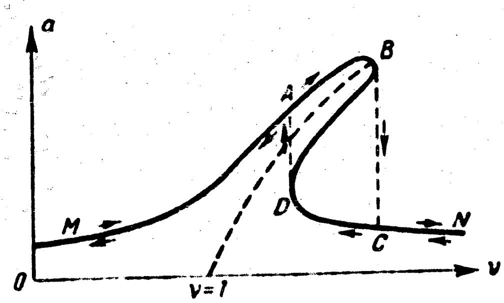

Fig. 60

in first approximation, will vary according to a cosine curve with the frequency of the external excitation, and at a constant amplitude and constant phase which can be determined, with an accuracy to terms of the second order of smallness, by the system of equations

$$\left.\begin{aligned} -\delta a - E_1\cos\theta &= 0, \\ a\left[\left(1+\frac{3a^2}{8}\right)^2-\nu^2\right]+E_1\sin\theta &= 0. \end{aligned}\right\} \qquad (14.32)$$

Then, eq.(14.5) will yield for eq.(14.28):

$$\delta_e(a) = \delta, \quad \omega_e(a) = 1+\frac{3a^2}{8}. \qquad (14.33)$$

By eliminating the phase θ from eq.(14.32) [or by directly substituting the values of $\delta_e(a)$ and $\omega_e(a)$ from eq.(14.33) in eq.(14.10)], we find the following relation between the amplitude of the stationary oscillations and the frequency of the external force:

$$a^2\left\{\left[\left(1+\frac{3a^2}{8}\right)^2-\nu^2\right]^2+\delta^2\right\} = E_1^2. \qquad (14.34)$$

from which we obtain

$$\nu = \sqrt{\omega_e^2(a) \pm \sqrt{\frac{E_1^2}{a^2} - \delta^2}}. \qquad (14.35)$$

By means of this relation we construct the curve of resonance (Fig.80), and also the skeleton curve determined by the equation

$$1 + \frac{3a^2}{8} = \nu \qquad (14.36)$$

(Fig.80, broken line).

From the resultant diagram, in accordance with the rule given previously, it is easy to establish the zones of stable and unstable amplitude.

Thus, the segments of the resonance curve MAB and DCN will correspond to the stable amplitudes. The points B and D will be points of discontinuity in the amplitude.

The diagram given in Fig.80 permits a complete analysis of the character of the oscillations in the system under study, with variations in the frequency of the external force. Thus, at increasing frequency of the external force, beginning with small values, the amplitude of the forced oscillations increases at first along the curve MAB. At the point B, a break in the amplitude takes place. The value of the amplitude passes with a jump to the point C, and then varies along the curve CN with a further increase in the frequency. If we now begin to decrease the frequency, the amplitude of the forced oscillation will vary along the curve NCD. When it reaches the point D, the amplitude will shift to the point A and will then vary along the upper branch of the resonance curve AM.

We note that the variations in frequency of the external force are considered to vary so slowly that, in practice, the system may be considered as being stationary at any instant. This question will be discussed in more detail below, in connection with the phenomenon of passage through resonance.

We now present a solution of eq.(12.28) corresponding to the second approximation. According to eqs.(14.4) and (13.42), we have, in second approximation,

$$x = a\cos(\nu t + \vartheta) + \frac{a^3}{32}\cos 3(\nu t + \vartheta), \qquad (14.37)$$

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where a and θ must be determined from the system of equations of second approximation

$$\frac{da}{dt} = -\frac{\delta a}{2} + \frac{3a^3\delta}{16} - E_1\left[\frac{1}{1+\nu} - \frac{3a^2(7-\nu)}{8(3-\nu)(1+\nu)^2}\right]\cos\theta - \frac{E_1\delta}{2(1+\nu)^2}\sin\theta, \tag{14.38}$$

$$\frac{d\theta}{dt} = 1 - \nu + \frac{3a^2}{8} - \frac{\delta^2}{8} - \frac{15a^4}{256} + \frac{E_1}{a}\left[\frac{1}{1+\nu} - \frac{3a^2(5-3\nu)}{8(3-\nu)(1+\nu)^2}\right]\sin\theta - \frac{E_1\delta}{2a(1+\nu)^2}\cos\theta.$$

As follows from eq.(14.37), higher harmonics appear in the second approximation, and the oscillation will no longer be purely sinusoidal.

By equating the right sides of eq.(14.38) to zero and eliminating the angle θ, we obtain the relation between the amplitude of the oscillations a and the frequency

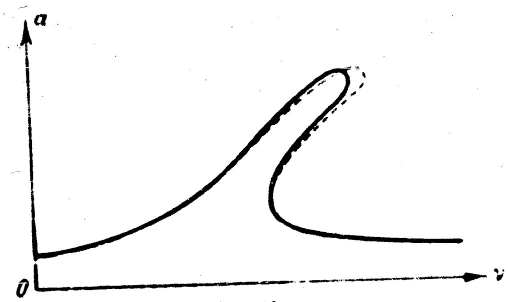

Fig.81

of the external force ν in second approximation. This relation will now be used for constructing the curve of resonance in second approximation (Fig.81, broken line).

As indicated above, in an oscillatory system of the type described by eq.(14.1), only one resonance, namely the principal resonance ($p = 1$, $q = 1$) can be detected, in first approximation. The submultiple resonances can be detected only on consideration of higher approximations.

To illustrate this, let us construct the first and second approximations for the oscillatory system described by eq.(14.28) for the case of $p = 1$, $q = 3$.

According to eqs.(13.39), (13.40), we have, in first approximation,

$$x = a\cos\left(\frac{1}{3}\nu t + \theta\right), \tag{14.39}$$

where a and θ must be determined from the equations

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$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\delta a}{2}, \\ \frac{d\theta}{dt} &= 1 - \frac{1}{3}\nu + \frac{3a^2}{8}. \end{aligned}\right\} \qquad (14.40)$$

The right sides of eq.(14.40) depend only on a and characterize the system in the nonresonant case. By integrating these equations, we obtain the following expressions for x:

$$x = a_0 e^{-\frac{\delta}{2}t} \cos\left[t - \frac{3a_0^2}{8\delta} e^{-\delta t} + \theta_0\right], \qquad (14.41)$$

where a_0 and θ_0 are arbitrary constants. Thus, in first approximation, the oscillations of the system are described by damping according to the exponential cosine law, while the frequency of the oscillations depends on the amplitude.

There will be no resonance effect in first approximation; however, in view of the fact that the amplitude of the external sinusoidal excitation is of the order ε, not even forced oscillations with the excitation frequency will be detected in first approximation (the forced oscillations will be perceptible on consideration of the refined first approximation).

Let us now calculate the second approximation: Making use of eqs.(13.41) and (13.42) we find the following expression for x:

$$x = a\cos\left(\frac{1}{3}\nu t + \theta\right) + \frac{a^3}{32}\cos 3\left(\frac{1}{3}\nu t + \theta\right) - \frac{E_1}{8}\sin\nu t, \quad 14.42)$$

in which a and θ are solutions of the equations

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\delta a}{2} + \frac{3}{16}\delta a^3 - \frac{3a^2E_1}{32\left(1+\frac{\nu}{3}\right)}\cos 3\theta, \\ \frac{d\theta}{dt} &= 1 - \frac{\nu}{3} + \frac{3a^2}{8} - \frac{\delta^2}{8} - \frac{15a^4}{256} + \frac{3aE_1}{32\left(1+\frac{\nu}{3}\right)}\sin 3\theta. \end{aligned}\right\} \qquad (14.43)$$

The expressions (14.42) and (14.43) testify to the influence of the external excitation on the oscillatory system which we detect on considering the second approximation. Thus, in accordance with eq.(14.42), the expression for x contains not only overtones of the natural frequency but also harmonics with the frequency of the

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external force. From eq.(14.43) we may detect the resonant zones and construct the resonance curves.

On equating the right sides of eq.(14.43) to zero, we obtain with an accuracy to terms of the third order of smallness, the following relations defining the stationary values of the amplitude a and the phase θ of the oscillations:

$$\left.\begin{aligned} -\delta_e(a)\,a - \frac{3a^2E_1}{32}\cos 3\theta = 0,\\ \omega_e^2(a) - \frac{\nu^2}{9} + \frac{3aE_1}{32}\sin 3\theta = 0. \end{aligned}\right\} \qquad (14.44)$$

Here the following notation has been introduced:

$$\left.\begin{aligned} \delta_e(a) = \delta - \frac{3}{8}\delta a^2,\\ \omega_e^2(a) = 1 + \frac{3a^2}{4} - \frac{\delta^2}{4} - \frac{15a^4}{128}. \end{aligned}\right\} \qquad (14.45)$$

On eliminating the phase θ from eq.(14.44), we find the relation between the amplitude and the frequency of the disturbing force:

$$\left[\omega_e^2(a) - \frac{\nu^2}{9}\right]^2 + \delta_e^2(a) = \frac{9a^2E_1^2}{1026}, \qquad (14.46)$$

or

$$\nu = 3\sqrt{\omega_e^2(a) \pm \sqrt{\frac{9a^2E_1^2}{1026} - \delta_e^2(a)}}, \qquad (14.47)$$

from which we are able to construct the resonance curve.

We now present an example for which the submultiple resonance can be detected in first approximation.

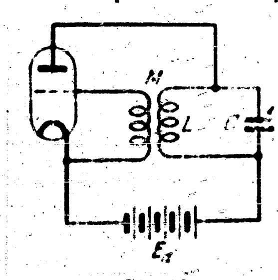

Fig.82

Consider the linear oscillatory circuit with feedback, using an electron tube (Fig.82).

It is well known that Mandel'shtam and Papaleksi (Bibl.27) studied, in this example, the influence of resonance of the n^{th} kind and obtained the solution of the resultant equation, for the steady state, by the Poincaré method, the van der Pol method was used for studying the process of build-up of the oscillations.

For this oscillatory system, the differential equation describing the motion has the form

$$CL\frac{d^2i}{d\tau^2} + CR\frac{di}{d\tau} + i = i_a + C\frac{d\mathscr{E}}{d\tau}, \qquad (14.48)$$

where

$$i_a = f_0(V_s) \qquad (14.49)$$

is the equation of the tube characteristic depending on the control voltage.

After a number of transformations, eq.(14.48) may be reduced to the form*

$$\frac{d^2x}{dt^2} + x = \varepsilon f\left(x, \frac{dx}{dt}\right) + E\sin nt, \qquad (14.50)$$

using the following notation:

$$\left.\begin{aligned} f\left(x, \frac{dx}{dt}\right) &= F'(x)\frac{dx}{dt} + \frac{\xi}{1+\xi}x, \\ F(x) &= \frac{1}{1+\xi}f_1(x) - 2\vartheta x, \end{aligned}\right\} \qquad (14.51)$$

$$\left.\begin{aligned} &\frac{dI}{d\tau} = x, \quad -\frac{nM_r}{L_0}\frac{E_0}{V_0} = E, \quad t = \frac{\tau}{\tau_0}, \\ &2\vartheta = \frac{nR}{\omega L}, \quad \xi = \frac{\omega^2 - n^2\omega_0^2}{n^2\omega_0^2}, \\ &I_a = \frac{i_a}{I_0} = \frac{f_0(V_s)}{I_0} = f_1\left(\frac{dI}{d\tau}\right). \end{aligned}\right\} \qquad (14.52)$$

Let us next discuss the resonant case. To be able to apply the formulas of Section 13 to the construction of the approximate solution, the following substitution of variables must be performed in eq.(14.50):

$$x = y + \frac{E}{1-n^2}\sin nt, \qquad (14.53)$$

after which we obtain the following equation:

$$\frac{d^2y}{dt^2} + y = \varepsilon f\left[y + \frac{E}{1-n^2}\sin nt, \frac{dy}{dt} + \frac{En}{1-n^2}\cos nt\right]. \qquad (14.54)$$

* Cf.Bibl.27, Vol.II, p.21

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Assume that the feedback in the circuit is effected by means of an electron tube having the characteristic

$$i_a = a + bx + cx^2 - dx^3, \tag{14.55}$$

where V_a = 12 volts and I_a = 142 ma, while a, b, c, and d are constants. Then we obtain the following expression for the right side of eq.(14.50):

$$f\left(x, \frac{dx}{dt}\right) = (k + 2x + \gamma x^2)\frac{dx}{dt} + \frac{\xi}{0{,}016}x, \tag{14.56}$$

using the values:

$$\left.\begin{aligned} &\varepsilon = \frac{0{,}016}{1+\xi}, \quad k = k_0 + 2\vartheta\frac{\xi}{\beta}, \quad \beta = 0{,}016, \\ &\vartheta = 0{,}013, \quad \gamma = 2, \quad k_0 = 0{,}05 \\ &(a = 0{,}95, \quad b = 3{,}35, \quad c = 2{,}25, \quad d = 1{,}5). \end{aligned}\right\} \tag{14.57}$$

On substituting the values of $f(x, \frac{dx}{dt})$ from eq.(14.56) in eq.(14.54), we find the following expression for the right side of this equation:

$$\begin{aligned} &f\left[y + \frac{E}{1-n^2}\sin nt, \ \frac{dy}{dt} + \frac{En}{1-n^2}\cos nt\right] = \\ &= \left[k + 2y + \frac{2E}{1-n^2}\sin nt + \gamma\left(y^2 + \frac{2yE}{1-n^2}\sin nt + \right.\right. \\ &\left.\left. + \frac{E^2}{(1-n^2)^2}\sin^2 nt\right)\right]\left(\frac{dy}{dt} + \frac{En}{1-n^2}\cos nt\right) + \\ &+ \frac{\xi}{0{,}016}\left(y + \frac{E}{1-n^2}\sin nt\right). \end{aligned} \tag{14.58}$$

Let us now construct the solution of eq.(14.54), in first approximation, for the case n = 2, i.e., for the case when a half-resonance may arise in the oscillatory system.

Making use of eqs.(13.39) and (13.35), and putting p = 1, q = 2, we obtain, after several calculations

$$y = a\cos(t + \theta), \tag{14.59}$$

where a and θ must be determined from the system of equations of first approximation

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon\left\{\frac{1}{2}a\left(k + \frac{\gamma a^2}{4}\right) + \frac{\gamma E^2 a}{36} + \frac{aE}{6}\sin 2\theta\right\}, \\ \frac{d\theta}{dt} &= \varepsilon\left\{-\frac{\xi}{2\varepsilon} + \frac{E}{6}\cos 2\theta\right\}. \end{aligned}\right\} \tag{14.60}$$

The systems of equations of first approximation (14.60) permits an investigation of both the stationary state and the build-up process of oscillations in resonance of the second kind.

To investigate the build-up of oscillations, the system (14.60) must be integrated, and a and θ found as functions of time. In this case, the integration of the system (14.60) can be completed. For this purpose, we perform the substitution of variables in eq.(14.60), according to the formulas:

$$u = a\cos\theta, \quad v = a\sin\theta. \tag{14.61}$$

After several calculations, we obtain, instead of eq.(14.60), the following system for the new variables u and v:

$$\left.\begin{aligned} \frac{du}{dt} &= \varepsilon\left\{\frac{1}{2}u\left[k + \frac{7}{4}(u^2 + v^2)\right] + \frac{7E^2}{36}u + \frac{E}{6}v + \frac{\xi}{2\nu}v\right\}, \\ \frac{dv}{dt} &= \varepsilon\left\{\frac{1}{2}v\left[k + \frac{7}{4}(u^2 + v)\right] + \frac{7E^2}{36}v + \frac{E}{6}u - \frac{\xi}{2\nu}u\right\}. \end{aligned}\right\} \tag{14.62}$$

The system (14.62), as previously shown (Bibl.27), may be reduced to an equation of the Bernoulli type.

Indeed, on multiplying eq.(14.62) by v and u, respectively, and subtracting the second product from the first, we get

$$\begin{aligned} v\frac{du}{dt} - u\frac{dv}{dt} &= v^2\frac{d}{dt}\left(\frac{u}{v}\right) = \\ &= \frac{\varepsilon}{2}\left[-(u^2 - v^2)\frac{E}{3} - (u^2 + v^2)\frac{\xi}{\nu}\right]. \end{aligned} \tag{14.63}$$

On multiplying the first equation of the system (14.62) by u and adding the result so obtained to the second equation multiplied by v, we obtain:

$$\frac{da}{dt} = \varepsilon a\left[k + \frac{7}{4}\left(a + \frac{2E^2}{9}\right)\right] + \frac{2\varepsilon uv}{3}E. \tag{14.64}$$

Setting

$$\frac{u}{v} = \chi, \tag{14.65}$$

the following system may be written instead of eqs.(14.63) and (14.64):

$$\frac{d\chi}{dt} = \frac{\varepsilon}{2}\left[\left(-\frac{E}{3} - \frac{\xi}{\nu}\right)\chi^2 - \left(-\frac{E}{3} + \frac{\xi}{\nu}\right)\right], \tag{14.66}$$

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$$\frac{da}{dt} = \varepsilon a\left[k + \frac{\gamma}{4}\left(a + \frac{2E^2}{9}\right)\right] - \frac{2\varepsilon E a}{3}\,\frac{\chi}{1+\chi^2}. \qquad (14.67)$$

Equation (14.66) is easily integrated.

After having determined $\chi(t)$ from it, eq.(14.67) may be reduced to the form:

$$\frac{da}{dt} = \frac{\varepsilon\gamma}{4}a^3 + \varphi(t)\,a, \qquad (14.68)$$

where $\varphi(t)$ is a known function of time.

By substitution of $W = \frac{1}{a}$, eq.(14.68) is reduced to the linear equation

$$\frac{dW}{dt} = -\varphi(t)\,W - \frac{\varepsilon\gamma}{4}. \qquad (14.69)$$

As a result we obtain the following well-known formula expressing the law of variation of the amplitude of oscillation with time:

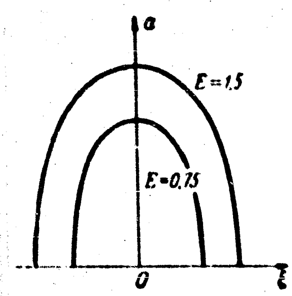

Fig.83

$$a = \frac{e^{\int_0^t \varphi(t)\,dt}}{C_1 - \frac{\varepsilon\gamma}{4}\int_0^t e^{\int_0^t \varphi(t)\,dt}\,dt}, \qquad (14.70)$$

where C_1 is an integration constant.

Let us now determine the steady oscillations, at constant amplitude and phase.

On equating the right sides of the system (14.60) to zero, we obtain the relations

$$\left.\begin{aligned} k + \frac{\gamma a^2}{4} + \frac{\gamma E^2}{18} + \frac{E}{3}\sin 2\theta &= 0, \\ -\frac{\xi}{\beta} + \frac{E}{3}\cos 2\theta &= 0, \end{aligned}\right\} \qquad (14.71)$$

determining the stationary values of the amplitudes and phase of the oscillations. By eliminating the phase θ from eq.(14.71), we find the well-known relation

$$a^2 = -\frac{2E^2}{9} - \frac{4}{\gamma}\left[k \pm \sqrt{\frac{E^2}{9} - \frac{\xi^2}{\beta^2}}\right], \qquad (14.72)$$

by means of which we can construct the resonance curves characterizing the relation of the amplitude a to the detuning ξ (Fig.83). The stationary values of the phase θ are found by the aid of the formula

$$\operatorname{tg} 2\theta = -\frac{k + \frac{\gamma a^2}{4} + \frac{\gamma E^2}{18}}{\frac{\xi}{E}}, \qquad (14.73)$$

where a is determined from eq.(14.72).

For determining the stable values of the stationary amplitude, we proceed according to the general rules.

We find first the quantities

$$\left.\begin{aligned} A_a'(a, \theta) &= \frac{1}{2}\left(k + \frac{3\gamma a^2}{4}\right) + \frac{\gamma E^2}{36} + \frac{E}{6}\sin 2\theta, \\ A_\theta'(a, \theta) &= \frac{Ea}{3}\cos 2\theta, \\ B_a'(a, \theta) &= 0, \\ B_\theta'(a, \theta) &= -\frac{E}{3}\sin 2\theta. \end{aligned}\right\} \qquad (14.74)$$

After this we can set up the equations of variation:

$$\left.\begin{aligned} \frac{d\delta a}{dt} &= \varepsilon\left\{\frac{1}{2}\left(k + \frac{3\gamma a^2}{4}\right) + \frac{\gamma E^2}{36} + \frac{E}{6}\sin 2\theta\right\}\delta a + \\ &\qquad\qquad + \frac{Ea}{3}\cos 2\theta\,\delta\theta, \\ \frac{d\delta\theta}{dt} &= -\frac{E}{3}\sin 2\theta\,\delta\theta, \end{aligned}\right\} \qquad (14.75)$$

from which we find the conditions of stability of the stationary values of a and θ:

$$\left.\begin{aligned} &\frac{1}{2}\left(k + \frac{3\gamma a^2}{4}\right) + \frac{\gamma E^2}{36} + \frac{E}{6}\sin 2\theta - \frac{E}{3}\sin 2\theta < 0, \\ &\left\{\frac{1}{2}\left(k + \frac{3\gamma a^2}{4}\right) + \frac{\gamma E^2}{36} + \frac{E}{6}\sin 2\theta\right\}\left(-\frac{E}{3}\sin 2\theta\right) > 0. \end{aligned}\right\} \qquad (14.76)$$

These conditions, after several transformations, may be represented in the form of the following well-known inequalities:

$$k + \frac{1}{2}\gamma a^2 + \frac{\gamma E^2}{18} < 0, \qquad (14.77)$$

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$$\gamma\left[k+\frac{\gamma a^2}{4}+\frac{\gamma E^2}{18}\right]>0, \qquad (14.78)$$

the analysis of which, together with the relation (14.72), permits determining the magnitude and boundaries of the stability zone of the periodic solution with the period 2π*.

Section 15. Influence of a Sinusoidal Force on a Nonlinear System with a Characteristic Composed of Rectilinear Segments

As our second special case, consider the oscillations in a nonlinear system with a characteristic composed of several rectilinear segments, the system being under the influence of a sinusoidal disturbing force.

Oscillatory systems for which the nonlinear restoring force has a character-

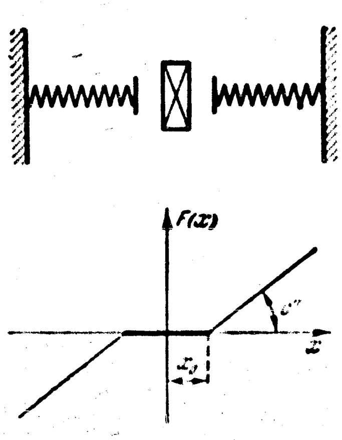

Fig.84

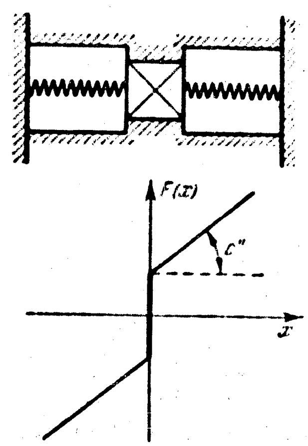

Fig.85

istic composed of segments of straight lines (Figs.84, 85, 86, 87), are widely met in technology.

A number of works have been devoted to the study of forced oscillations in nonlinear systems of this type, for example the reports published by A.I.Lur'ye and

* Cf. Bibl.(27), Vol.88, p.36

A.I.Chekmarev (Bibl.51), where the solution is derived by Galerkin's method; for certain initial conditions, of special form, Den Hartog (Bibl.52) constructed a solution which may be considered as exact, but which is exceedingly unwieldy.

For the solution of such problems it seems more convenient to use the above asymptotic method which, in first approximation, yields the same results as the Galerkin method, but at the same time makes it easy to determine the second approxi-

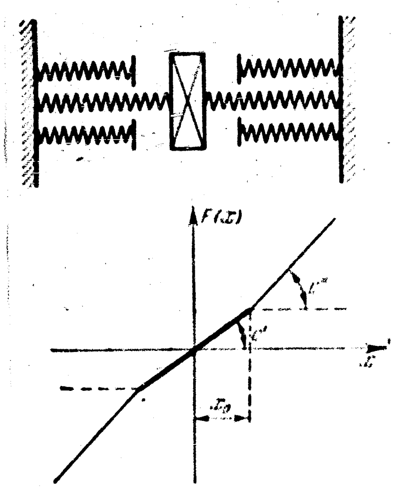

Fig.86

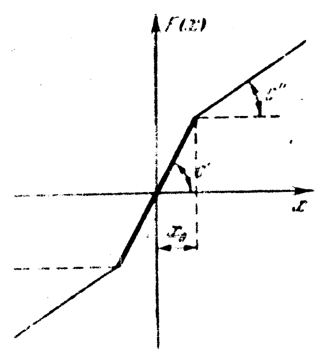

Fig.87

mation, to find a correction to the frequency of the second approximation, and not only to approximate the stationary state but to follow the motion in the system during the period when the oscillations are building up.

Thus, let us assume that the oscillations of the system are described by an equation of the form

$$\frac{d^2x}{dt^2} + F(x) = \varepsilon f_1\left(\frac{dx}{dt}\right) + \varepsilon E \sin \nu t, \qquad (15.1)$$

where the function F(x), expressing the relation between the nonlinear restoring force and the displacement, is an odd function of x (the case of a symmetrical non-

linear characteristic) and has the form, for instance, given in Figs. 84-87.

We note that, if the characteristic of the nonlinear restoring force is asymmetric, the solution can likewise be constructed without trouble by the aid of the above method.

Assume that F(x) may be written in the form

$$F(x) = c''x + \varepsilon f(x); \tag{15.2}$$

Then, instead of eq. (15.1), we may consider the following:

$$\frac{d^2x}{dt^2} + c''x = -\varepsilon f(x) + \varepsilon f_1\left(\frac{dx}{dt}\right) + \varepsilon E \sin \nu t, \tag{15.3}$$

Consequently, eqs. (14.2) and (14.3), in first approximation, will yield the solution

$$x = a \cos(\nu t + \theta), \tag{15.4}$$

where a and θ must be determined from the system of equations

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f_1(-a\omega \sin\psi)\sin\psi\, d\psi - \frac{\varepsilon E}{\omega+\nu}\cos\theta, \\ \frac{d\theta}{dt} &= \omega - \nu + \frac{\varepsilon}{2\pi\omega a}\int_0^{2\pi} f(a\cos\psi)\cos\psi\, d\psi + \frac{\varepsilon E}{a(\omega+\nu)}\sin\theta. \end{aligned}\right\} \tag{15.5}$$

Bearing in mind that

$$\omega^2 + \frac{\varepsilon}{\pi a}\int_0^{2\pi} f(a\cos\psi)\cos\psi\, d\psi = \frac{1}{\pi a}\int_0^{2\pi} F(a\cos\psi)\cos\psi\, d\psi, \tag{15.6}$$

we transform eq. (15.5) into the form

$$\left.\begin{aligned} \frac{da}{dt} &= -\delta_e(a)\, a - \frac{\varepsilon E}{\omega+\nu}\cos\theta, \\ \frac{d\theta}{dt} &= \omega_e(a) - \nu + \frac{\varepsilon E}{a(\omega+\nu)}\sin\theta, \end{aligned}\right\} \tag{15.7}$$

where, as above,

$$\delta_e(a) = \frac{\varepsilon}{2\pi\omega a}\int_0^{2\pi} f_1(-a\omega\sin\psi)\sin\psi\, d\psi, \tag{15.8}$$

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$$\omega_e^2(a) = \frac{1}{\pi a}\int_0^{2\pi} F(a\cos\psi)\cos\psi\, d\psi. \tag{15.9}$$

On equating the right sides of eq.(15.7) to zero, and eliminating θ, we find the relation between a and ν for the stationary state:

$$a^2[(\omega_e^2(a)-\nu^2)^2 + 4\nu^2\delta_e^2(a)] = \varepsilon^2 E^2. \tag{15.10}$$

If we neglect friction, then instead of eq.(15.10) we obtain the following simple formula:

$$a[\omega_e^2(a)-\nu^2] = \pm\varepsilon E, \tag{15.11}$$

where, in the right-hand side, the plus sign must be used for a > 0 and the minus sign for a < 0.

We now present the corresponding formulas for the second approximation:

Neglecting friction in eq.(15.1), we have

$$x = a\cos(\nu t+\theta) + \varepsilon u_1(a, \nu t, \nu t+\theta), \tag{15.12}$$

where

$$u_1(a, \nu t, \nu t+\theta) = \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{\cos n(\nu t+\theta)}{\omega^2(1-n^2)}\int_0^{2\pi} f(a\cos\psi)\cos\psi\, d\psi, \tag{15.13}$$

while the amplitude of the stationary oscillations is determined by the relation

$$a[\omega_e^2(a)-\nu^2] + \frac{1}{2}\sum_{n=2}^{\infty}\frac{f_n(a)[f_{n+1}^{(1)}(a)+f_{n-1}^{(1)}(a)]}{\omega^2(1-n^2)} = \pm\varepsilon E, \tag{15.14}$$

in which the following symbols are used:

$$\left.\begin{aligned} f_n(a) &= \frac{1}{\pi}\int_0^{2\pi} f(a\cos\psi)\cos n\psi\, d\psi, \\ f_n^{(1)}(a) &= \frac{1}{\pi}\int_0^{2\pi} f_a'(a\cos\psi)\cos n\psi\, d\psi. \end{aligned}\right\} \tag{15.15}$$

We pass now to the construction of the resonance curves. For the characteristics of the nonlinear restoring force, given in Figs.84-87, we have

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$$F(x) = \begin{cases} c'x & -x_0 \leqslant x \leqslant x_0, \\ c''x + (c' - c'')x_0 & x_0 \leqslant x \leqslant \infty, \\ c''x - (c' - c'')x_0 & -\infty \leqslant x \leqslant -x_0. \end{cases} \qquad (15.16)$$

Let us put $a > 0$, $a > x_0$, and let us denote by ψ_0 the smallest root of the equation

$$x_0 = a \cos \psi. \qquad (15.17)$$

Then it becomes obvious that

$$\varepsilon f(a \cos \psi) = \begin{cases} (c' - c'') a \cos \psi & \text{for} \quad \psi_0 \leqslant \psi \leqslant \pi - \psi_0, \\ (c' - c'') a \cos \psi_0 & \text{for} \quad 0 \leqslant \psi \leqslant \psi_0, \\ -(c' - c'') a \cos \psi_0 & \text{for} \quad \pi - \psi_0 \leqslant \psi \leqslant \pi. \end{cases} \qquad (15.18)^*$$

On dissecting the interval of integration into three parts, we find, after elementary calculations:

$$\frac{1}{\pi} \int_0^{2\pi} \varepsilon f(a \cos \psi) \cos \psi \, d\psi = \\ = c''a + \frac{2}{\pi}(c' - c'')\left[a \arcsin \frac{x_0}{a} + x_0 \sqrt{1 - \left(\frac{x_0}{a}\right)^2}\right]. \qquad (15.19)^*$$

For $a < 0$, $|a| > x_0$, after analogous calculations, we have

$$\frac{1}{\pi} \int_0^{2\pi} \varepsilon f(a \cos \psi) \cos \psi \, d\psi = \\ = c''a - \frac{2}{\pi}(c' - c'')\left[a \arcsin \frac{x_0}{a} + x_0 \sqrt{1 - \left(\frac{x_0}{a}\right)^2}\right]. \qquad (15.20)^*$$

In this way, in the absence of friction, eq. (15.11) will give the following relation between the amplitude of stationary oscillations and the frequency of the external force:

$$a(c'' - \nu^2) + \frac{2}{\pi}(c' - c'')\left[x_0 \sqrt{1 - \left(\frac{x_0}{a}\right)^2} + a \arcsin \frac{x_0}{a}\right] = \pm \varepsilon E$$

---

* Translator's note: See errata sheet.

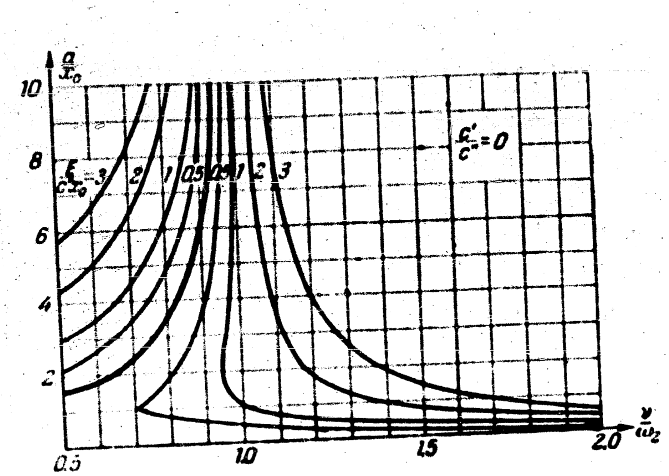

Fig. 88

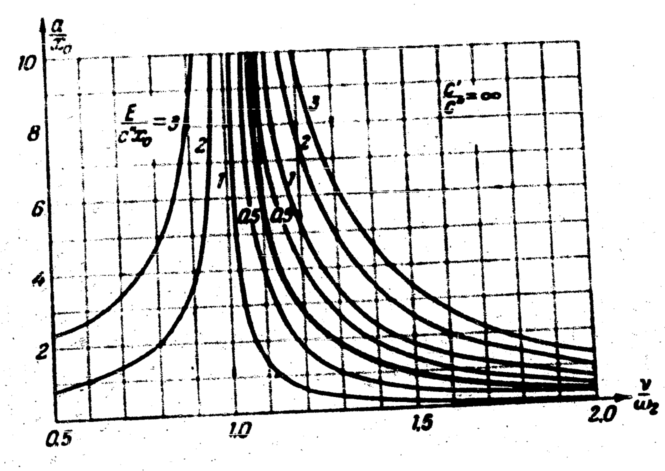

Fig. 89

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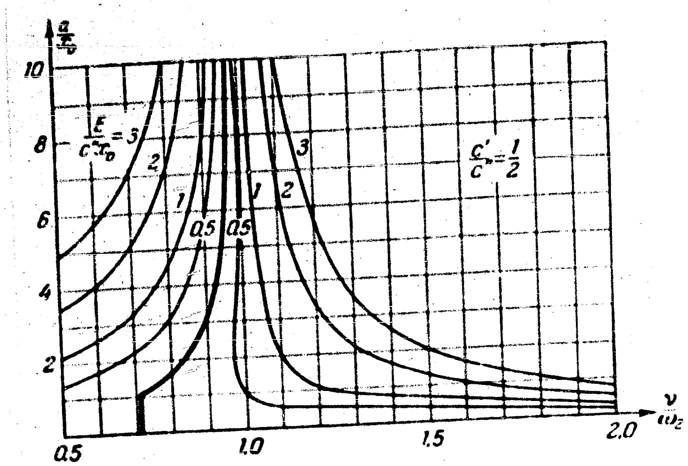

Fig. 90

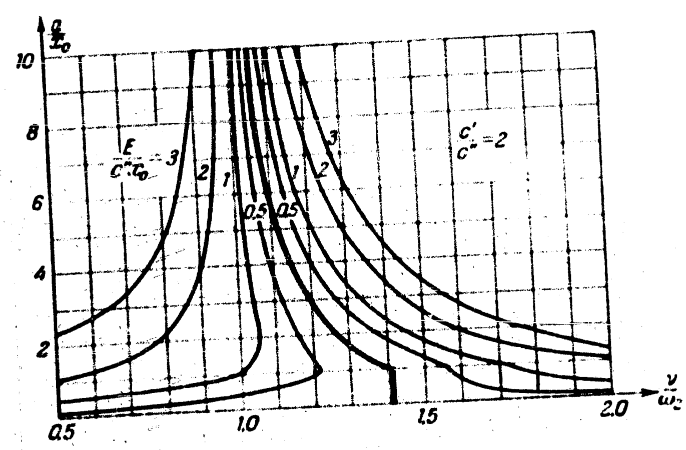

Fig. 91

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or, denoting $\left|\frac{a}{x_0}\right| = A$,

$$A(c'' - \nu^2) + \frac{2}{\pi}(c' - c'')\left[\sqrt{1 - \frac{1}{A^2}} + A \arcsin \frac{1}{A}\right] = \pm \frac{\varepsilon E}{x_0}. \quad (15.21)$$

Using the relation (15.21), let us construct the family of resonance curves (Figs. 88-91) for various values of $\frac{E}{c'' x_0}$.

The resonance curves presented in these diagrams practically coincide (within the limits of accuracy of the construction of the graphs) with the resonance curves constructed by the exact formulas of Den Hartog and Gels.

For constructing the resonance curve in second approximation, the expression for the sum entering into eq.(15.14) must be determined. We note that the calculation of the sum involves no difficulty since its summands rapidly decrease with increasing n; therefore, it is sufficient to calculate only the first summands.

Taking eq.(15.18) into consideration, we find the following values for the integrals of eq.(15.15):

$$\frac{1}{\pi}\int_0^{2\pi} \varepsilon f(a\cos\psi)\cos n\psi\, d\psi =$$

$$= \frac{(c' - c'')}{\pi} a \left\{ \frac{2x_0}{na}(-1)^{\frac{n-1}{2}} \cos\left(n \arcsin \frac{x_0}{a}\right) + \right.$$

$$+ \frac{(-1)^{\frac{n-1}{2}}}{n-1} \sin\left[(n-1)\arcsin\frac{x_0}{a}\right] +$$

$$\left. + \frac{(-1)^{\frac{n+1}{2}}}{n+1} \sin\left[(n+1)\arcsin\frac{x_0}{a}\right]\right\} \quad (15.22)^*$$

$$(n = 3, 5, 7, 9, \ldots),$$

$$\frac{1}{\pi}\int_0^{2\pi} \varepsilon f'_a(a\cos\psi)\cos n\psi\, d\psi = -\frac{2}{\pi}(c' - c'')\sin n \arccos\frac{x_0}{a}$$

$$(n = 2, 4, 6, 8, \ldots).$$

On substituting eq.(15.22) in eq.(15.14) and confining ourselves in the sum to five summands, we obtain

* Translator's note: See errata sheet.

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$$
a(\nu^2 - c'') - \frac{2}{\pi}(c' - c'')\left[x_0\sqrt{1 - \left(\frac{x_0}{a}\right)^2} + a \arcsin\frac{x_0}{a}\right] +
$$

$$
+ \frac{1}{2\omega^2}\left\{\frac{f_3(a)\,[f_2^{(1)}(a) + f_4^{(1)}(a)]}{8} + \frac{f_5(a)\,[f_4^{(1)}(a) + f_6^{(1)}(a)]}{24}\right\} + \ldots
$$

$$
\ldots + \varepsilon E = 0, \tag{15.23}
$$

where $f_n(a)$ and $f_n^{(1)}(a)$ $(n = 2, 3, 4, 5, 6)$ are determined by the expressions:

$$
f_3(a) = -\frac{4(c' - c'')}{3\pi}\frac{x_0}{a}\sqrt{1 - \frac{x_0^2}{a^2}}\left[1 + 2\frac{x_0^2}{a^2}\right]a,
$$

$$
f_5(a) = -\frac{4(c' - c'')}{15\pi}\frac{x_0}{a}\sqrt{1 - \frac{x_0^2}{a^2}}\left[3 - 11\frac{x_0^2}{a^2} + 8\frac{x_0^4}{a^4}\right]a,
$$

$$
f_2^{(1)}(a) + f_4^{(1)}(a) = -\frac{8(c' - c'')}{\pi}\frac{x_0}{a}\sqrt{1 - \frac{x_0^2}{a^2}}\left[1 - \frac{x_0^2}{a^2}\right],
$$

$$
f_4^{(1)}(a) + f_6^{(1)}(a) = -\frac{4(c' - c'')}{\pi}\left[\frac{1}{3}\left(1 - \frac{x_0^2}{a^2}\right)\left(1 - 4\frac{x_0^2}{a^2}\right) +\right.
$$

$$
\left. + \frac{x_0}{a}\sqrt{1 - \frac{x_0^2}{a^2}}\left(1 - 2\frac{x_0^2}{a^2}\right)\right]. \tag{15.24}
$$

Using the relation (15.23) we construct the resonance curves in second approximation (cf. Figs.92, 93, 94 where the resonance curves, together with the skeleton curves in second approximation, are indicated by broken lines. It should be noted that the scale here is considerably increased; the heavy lines are the curves in first approximation).

Figures 95-98 give the families of resonance curves constructed, allowing for friction according to eq.(15.10). In the construction of these curves, the friction was taken as proportional to the velocity, and $\frac{\delta}{c''} = 0.1$.

Let us also consider the case when the characteristic of the nonlinear restoring force has the form shown in Fig.99.

For calculating $\omega_e(a)$ from eq.(15.9), we have

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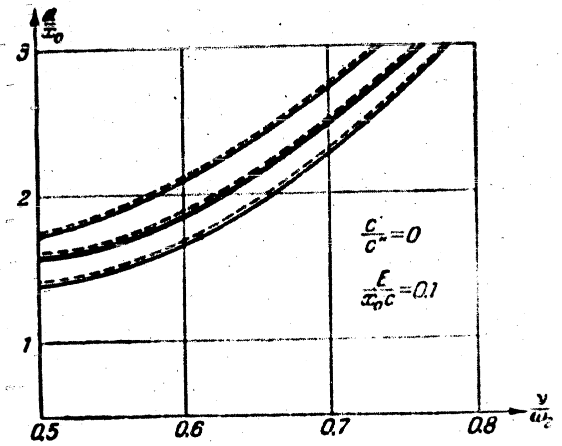
$\frac{a}{x_0}$
3
2
1
$\frac{c'}{c''}=0$
$\frac{E}{x_0 c}=0.1$
0.5
0.6
0.7
0.8
$\frac{\nu}{\omega_2}$

Fig.92

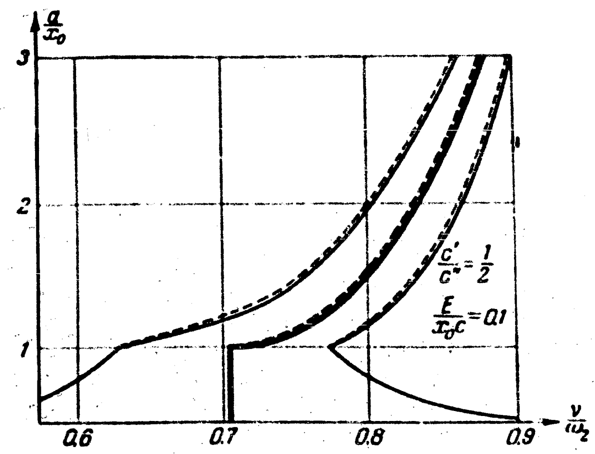
$\frac{a}{x_0}$
3
2
1
$\frac{c'}{c''}=\frac{1}{2}$
$\frac{E}{x_0 c}=0.1$
0.6
0.7
0.8
0.9
$\frac{\nu}{\omega_2}$

Fig.93

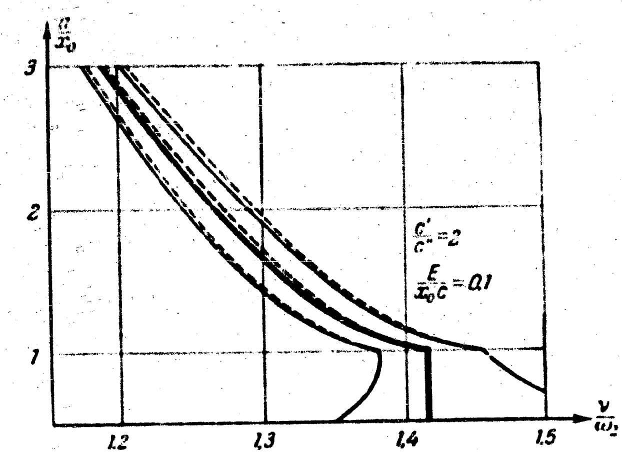

Fig. 94

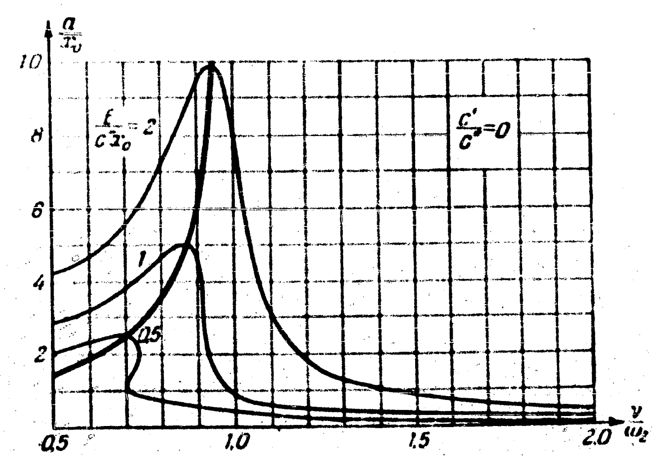

Fig. 95

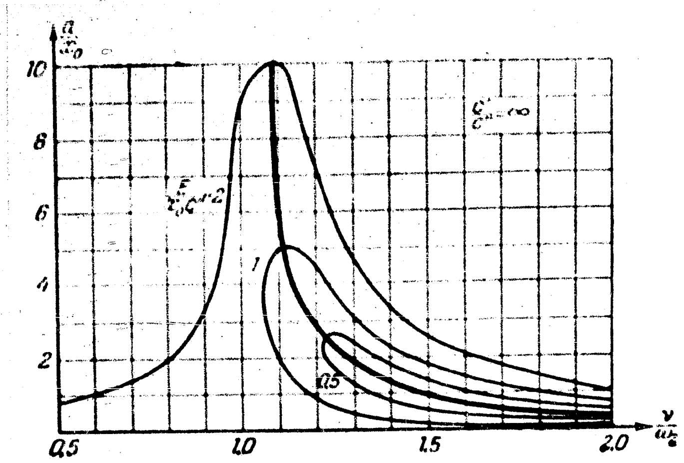

Fig. 96

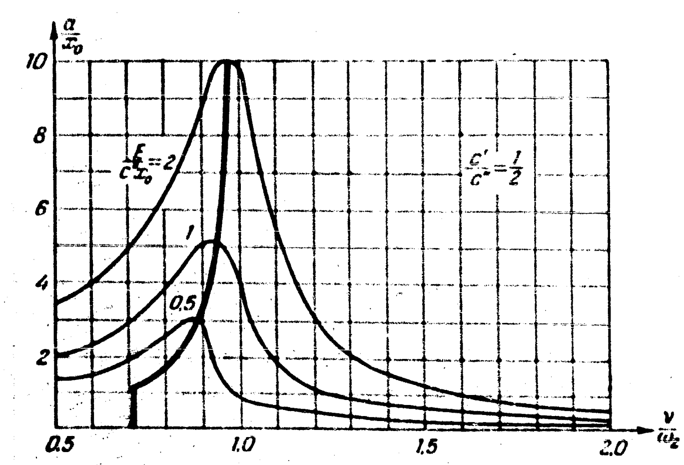

Fig. 97

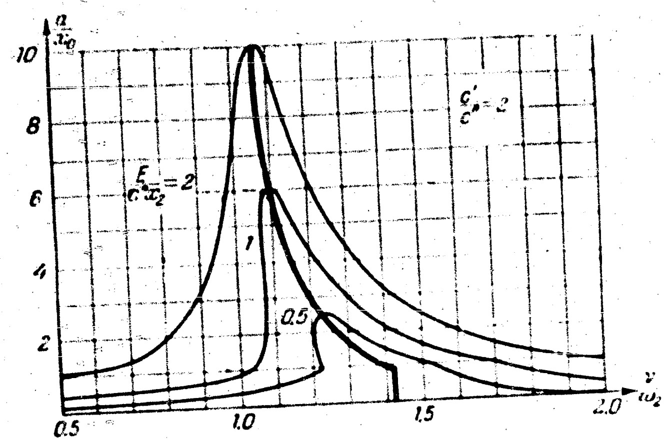

Fig. 98

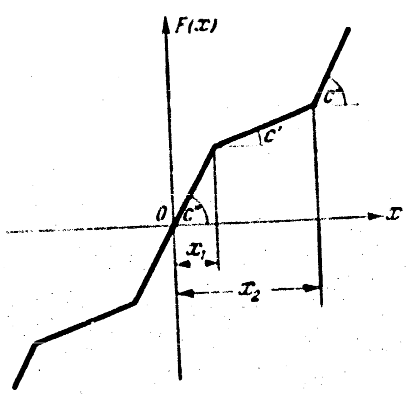

Fig. 99

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$$F(x)=\begin{cases} c''x & \text{for } -x_1\leqslant x\leqslant x_1,\\ c''x+(c''-c')(x_1-x) & \text{»} \quad x_1\leqslant x\leqslant x_2,\\ c''x+(c''-c')(x_1-x) & \text{»} \quad -x_2\leqslant x\leqslant -x_1,\\ c''x-(c''-c')(x_2-x_1) & \text{»} \quad x_2\leqslant x\leqslant \infty,\\ c''x+(c''-c')(x_2-x_1) & \text{»} \quad -\infty\leqslant x\leqslant -x_2.\end{cases} \tag{15.25}$$

To determine the integral on the right side of eq.(15.9), let us denote by ψ_1 and ψ_2 the smallest roots of the equations $x_1 = a\cos\psi$, $x_2 = a\cos\psi$. Then we may write :

$$F(a\cos\psi)=\begin{cases} c''a\cos\psi, \quad \psi_1\leqslant\psi\leqslant\pi-\psi_1,\\ c''a\cos\psi+(c''-c')(a\cos\psi_1-a\cos\psi),\\ \qquad \psi_2\leqslant\psi\leqslant\psi_1,\\ c''a\cos\psi+(c''-c')(a\cos\psi_1-a\cos\psi),\\ \qquad \pi-\psi_1\leqslant\psi\leqslant\pi-\psi_2,\\ c''a\cos\psi-(c''-c')(a\cos\psi_2-a\cos\psi_1),\\ \qquad 0\leqslant\psi\leqslant\psi_2,\\ c''a\cos\psi+(c''-c')(a\cos\psi_2-a\cos\psi_1),\\ \qquad \pi-\psi_2\leqslant\psi\leqslant\pi.\end{cases} \tag{15.26}$$

Then, by resolving the interval of integration into five parts, we obtain the following expression for $\omega_e^2(a)$:

$$\omega_e^2(a)=\omega^2+\frac{2(c''-c')}{\pi}\left\{\arccos\frac{x_2}{a}-\arccos\frac{x_1}{a}+\frac{x_1}{a}\sqrt{1-\frac{x_1^2}{a^2}}-\frac{x_2}{a}\sqrt{1-\frac{x_2^2}{a^2}}\right\}. \tag{15.27}$$

Assuming, for simplification, that the friction is proportional to the first power of the velocity $\varepsilon f_1(\frac{dx}{dt}) = -2\delta\frac{dx}{dt}$, we have:

$$\delta_e(a)=\delta.$$

For constructing the graph of the relation between the amplitude of the forced oscillations and the frequency of the external disturbing force, we obtain the following relation

$$\nu^2=\omega_e^2(a)\pm\sqrt{\frac{\varepsilon^2E^2}{a^2}-4\delta^2\omega^2}, \tag{15.28}$$

from which we can construct a graph determining the value of ν as a function of a. According to the relation

$$\nu = \omega_e(a) \tag{15.29}$$

let us construct the curve of the functional relationship of the natural frequency

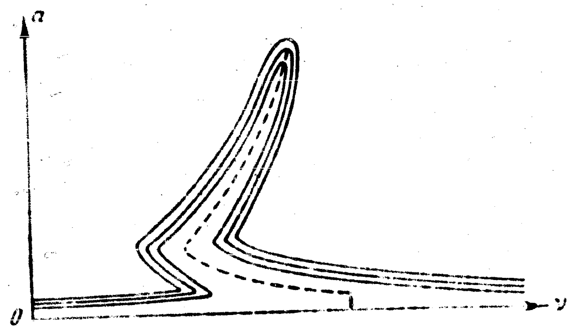

Fig. 100

and the amplitude. In Fig. 100 we give the family of curves characterizing the relationship between the amplitude and the frequency of the external force, at various amplitudes of the external force (family of resonance curves). The criteria given previously in the text readily permit determining the stable and unstable zones of these resonance curves, as well as the points of "discontinuity" and "jump" of the amplitude. Figure 101 gives one of the resonance curves, where the segments shown as heavy lines correspond to the stable amplitudes, and those shown as thin lines correspond to the unstable amplitudes.

Thus, according to the resonance curve of Fig. 101, an infinitely slow variation in the frequency of the external force (i.e., in a stationary state), beginning from small values, will cause the amplitude of the forced oscillations to increase first along the curve MA; from the point A, the amplitude jumps to the point B and then varies along the curve BC. At the point C, a discontinuity of the amplitude again occurs, the amplitude jumps to the point D and, on further increase in the frequency, varies along the curve DN.

If we now begin to decrease the frequency of the applied force, the amplitude

of the forced oscillation will vary along the curve NDE. From the point E, the amplitude jumps to the point F, and then varies along the curve FBG. When it reaches the point G, the amplitude will jump to the point H, after which it will vary along the curve HM. Here again we have to do with the hysteresis phenomena, characteristic for a nonlinear oscillatory system and analogous to those considered previously in the text.

These examples show the great flexibility and wide range of application of this method. The formulas obtained by us, even for a complex characteristic of the non-

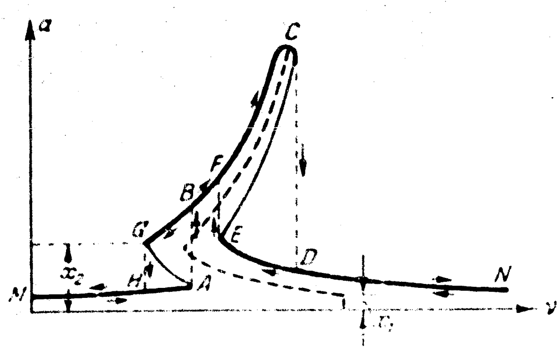

Fig. 101

linear restoring force, are rather simple. In the case where the characteristic is composed of two segments of straight lines, we obtain a very simple formula (we note that, in first approximation, the result for the stationary state corresponds with the result by Lur'ye and Chekmarev). If the characteristic of the nonlinear restoring force consists of several rectilinear segments, then our formulas undergo an entirely insignificant modification, while the Den Hartog method becomes so complicated as to become inapplicable to actual problems.

Section 16. Parametric Resonance

In the present Section we will consider certain oscillatory systems whose study

consists in an integration of differential equations with variable coefficients (as a function of time). The case of periodic coefficients is of greatest interest. It is well known that, in addition to problems of celestial mechanics, a number of purely technical problems lead to the consideration of differential equations with periodic coefficients*.

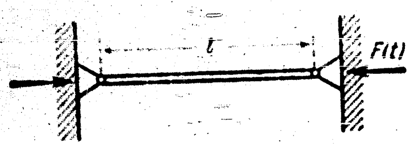

Fig. 102

One of the typical problems that lead to the consideration of these equations is the problem of the transverse oscillations of a rod under the action of longitudinal periodic forces.

Assume that a rod of the length l, attached by terminal hinges to an area of cross section A, rigidity EI, and density γ, is subjected to the periodic longitudinal force

$$F = F(t) \tag{16.1}$$

(see Fig. 102).

Then the differential equation of the transverse vibration of the rod may be represented in the following form

$$EI\frac{\partial^4 y}{\partial z^4} + \frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} + F(t)\frac{\partial^2 y}{\partial z^2} = 0, \tag{16.2}$$

where g is the acceleration of gravity.

In the case of the hinged ends, the boundary conditions for the differential equations (16.2) will be

$$\left.\begin{aligned} y|_{z=0} = 0, \quad \left.\frac{\partial^2 y}{\partial z^2}\right|_{z=0} = 0, \\ y|_{z=l} = 0, \quad \left.\frac{\partial^2 y}{\partial z^2}\right|_{z=l} = 0. \end{aligned}\right\} \tag{16.3}$$

Obviously, eq. (16.2) as well as the boundary conditions (16.3), by means of the substitution

* Cf. for example, V.N.Chelomey (Bibl. 43, 44).

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$$y = x \sin \pi \frac{z}{l} \tag{16.4}$$

may be reduced to the following:

$$\frac{d^2x}{dt^2} + \omega^2 \left[1 - \frac{l^2}{\pi EI} F(t) \right] x = 0, \tag{16.5}$$

where the notation

$$\omega^2 = \frac{g\pi^4 EI}{\gamma A l^4}. \tag{16.6}$$

is introduced.

Equation (16.5) is the well-known Hill equation.

The problem of oscillations of a mathematical pendulum whose axis of rotation performs an assigned periodic motion in a vertical direction, the problem of oscillations of a mechanical system with periodically varying rigidity, the problems of amplitude modulation, and many others, can be reduced to eq.(16.5).

In the case where the periodic function F(t) has the following form:

$$F(t) = P_0 \cos \nu t, \tag{16.7}$$

equation (16.5) is replaced by the equation

$$\frac{d^2x}{dt^2} + \omega^2 \left[1 - \frac{l^2 P_0}{\pi EI} \cos \nu t \right] x = 0, \tag{16.8}$$

which is called the Mathieu equation.

Both the Mathieu equation and the Hill equation are special cases of the second-order differential equation with periodic coefficients

$$\frac{d^2x}{dt^2} + p(t) \frac{dx}{dt} + q(t)x = 0, \tag{16.9}$$

where p(t) and q(t) are periodic functions of t with the period Ω.

Equations of the type of eq.(16.9) have been investigated by several scientists, but the existing theories [cf. for example, A.M.Lyapunov (Bibl.25)] allow merely a qualitative analysis of the behavior of the solutions of eq.(16.9), without indicating methods for constructing approximate solutions or methods for solving the question of the stability of these solutions.

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For the special case of eq.(16.9), for the Mathieu equation, solutions (the Mathieu functions) have been constructed, and an extensive literature is devoted to them.

In many cases differential equations with periodic coefficients may be reduced to eq.(12.1) discussed in Section 12, so that the approximate solutions may be constructed by means of the method set forth there.

In the following, we will construct approximate solutions and determine the stability zones in first and second approximation for the simplest case of an equation with periodic coefficients (16.9), that of the Mathieu equation; the result will then be compared with the solutions given in the literature.

Next, let us construct the approximate solutions for eq.(16.8), which we may write in the form

$$\frac{d^2x}{dt^2} + \omega^2(1 - h\cos\nu t)\,x = 0, \tag{16.10}$$

where we have put

$$h = \frac{P_0 l^2}{\pi E I} \ll 1.$$

As already stated, in first approximation we may consider, for an equation of the type of eq.(16.10) only the principal submultiple resonance p = 1, q = 2. Assuming that $\omega \approx \frac{\nu}{2}$, let us construct the approximate solutions corresponding to the resonant case.

In first approximation, making use of eq.(13.25), we have

$$x = a\cos\left(\frac{\nu}{2}t + \theta\right), \tag{16.11}$$

where a and θ must be determined from the systems of equations for

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{ah\omega^2}{2\nu}\sin 2\theta, \\ \frac{d\theta}{dt} &= \omega - \frac{\nu}{2} - \frac{h\omega^2}{2\nu}\cos 2\theta. \end{aligned}\right\} \tag{16.12}$$

To solve this system of equations of first approximation, let us introduce the new variables u and v according to the formulas

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$$u = \cos\vartheta, \qquad v = a\sin\vartheta. \tag{16.13}$$

Differentiating eq.(16.13) on the basis of eq.(16.12) will give

$$\left.\begin{aligned} \frac{du}{dt} &= \frac{da}{dt}\cos\vartheta - \frac{d\vartheta}{dt}a\sin\vartheta = \left[-\frac{h\omega^2}{2\nu} - \left(\omega - \frac{\nu}{2}\right)\right]a\sin\vartheta, \\ \frac{dv}{dt} &= \frac{da}{dt}\sin\vartheta + \frac{d\vartheta}{dt}a\cos\vartheta = \left[-\frac{h\omega^2}{2\nu} + \left(\omega - \frac{\nu}{2}\right)\right]a\cos\vartheta \end{aligned}\right\} \tag{16.14}$$

or

$$\left.\begin{aligned} \frac{du}{dt} &= \left[-\frac{h\omega^2}{2\nu} - \left(\omega - \frac{\nu}{2}\right)\right]v, \\ \frac{dv}{dt} &= \left[-\frac{h\omega^2}{2\nu} + \left(\omega - \frac{\nu}{2}\right)\right]u. \end{aligned}\right\} \tag{16.15}$$

Thus the equation of first approximation (16.12) has been reduced by us to a system of two linear equations with constant coefficients.

The character of the solutions of the system of equations (16.15) and, consequently, of the solutions of the system (16.12), depends on the roots of the characteristic equation

$$\begin{vmatrix} \lambda & +\frac{h\omega^2}{2\nu} + \left(\omega - \frac{\nu}{2}\right) \\ +\frac{h\omega^2}{2\nu} - \left(\omega - \frac{\nu}{2}\right) & \lambda \end{vmatrix} = 0$$

or

$$\lambda^2 - \frac{h^2\omega^4}{4\nu^2} + \left(\omega - \frac{\nu}{2}\right)^2 = 0. \tag{16.16}$$

Let us denote the roots of this equation by

$$+\lambda, \quad -\lambda,$$

where

$$\lambda = \sqrt{\frac{h^2\omega^4}{4\nu^2} - \left(\omega - \frac{\nu}{2}\right)^2}. \tag{16.17}$$

Then the general solution of the system of differential equations (16.15) may be presented in the following form:

$$u = C_1 e^{\lambda t} + C_2 e^{-\lambda t},$$
$$v = C_1 \frac{-\frac{h\omega^2}{2\nu} + \left(\omega - \frac{\nu}{2}\right)}{\lambda} e^{\lambda t} + C_2 \frac{\frac{h\omega^2}{2\nu} - \left(\omega - \frac{\nu}{2}\right)}{\lambda} e^{-\lambda t}, \tag{16.18}$$

where C_1, C_2 are arbitrary constants, determined from the initial conditions.

Let us now determine the amplitude a and the phase θ of the oscillations entering the right side of eq.(16.11). We have

$$\left.\begin{aligned} a^2 &= u^2 + v^2, \\ \theta &= \operatorname{arctg} \frac{v}{u}. \end{aligned}\right\} \tag{16.19}$$

From eqs.(16.17), (16.18), and (16.19) it is obvious that, at an imaginary λ, the amplitude a will be a bounded function of time.

In the case where λ is real, the amplitude a will increase by an exponential law. This case corresponds to the presence of a fundamental submultiple resonance in the system.

According to eq.(16.17), the condition for λ to be real is as follows:

$$\frac{h\omega^2}{2\nu} > \left|\omega - \frac{\nu}{2}\right|, \tag{16.20}$$

or, with an accuracy to terms of the first order of smallness,

$$\frac{h\omega}{4} > \left|\omega - \frac{\nu}{2}\right|,$$

since $\nu = 2\omega + O(h)$.

Thus, if the frequency of the external excitation is located within the interval

$$2\omega\left(1 - \frac{h}{4}\right) < \nu < 2\omega\left(1 + \frac{h}{4}\right), \tag{16.21}$$

a fundamental submultiple resonance will arise in the system; at this resonance, the amplitude of oscillations will increase by an exponential law. In view of the fact that this resonance results from the periodic variation of one of the parameters of the oscillatory system, it is often called parametric resonance.

The inequality (16.21) defines a zone of instability within which the position

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of equilibrium x = 0 is unstable and oscillations are self-excited in the system.

Let us now construct and analyze the second approximation.

According to eqs.(13.5), (13.21), (13.25) and (13.26), we have, in second approximation,

$$x = a\cos\left(\frac{\nu}{2}t+\vartheta\right) - \frac{ah\omega}{8\left(\omega+\frac{\nu}{2}\right)}\cos\left(\frac{3}{2}\nu t+\vartheta\right), \qquad (16.22)$$

where a and θ must be determined from the following systems of equations

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{ah\omega^2}{2\nu}\sin 2\vartheta, \\ \frac{d\vartheta}{dt} &= \omega-\frac{\nu}{2}+\frac{h^2(\omega+\nu)\omega}{32\left(\omega+\frac{\nu}{2}\right)}-\frac{h\omega^2}{2\nu}\cos 2\vartheta. \end{aligned}\right\} \qquad (16.23)$$

The system (16.23), by the substitution of variables (16.13), is likewise reduced to the system of linear equations

$$\left.\begin{aligned} \frac{du}{dt} &= \left[-\frac{h\omega^2}{2\nu}-\left(\omega-\frac{\nu}{2}\right)-\frac{h^2(\omega+\nu)\omega}{32\left(\omega+\frac{\nu}{2}\right)}\right]v, \\ \frac{dv}{dt} &= \left[-\frac{h\omega^2}{2\nu}+\left(\omega-\frac{\nu}{2}\right)+\frac{h^2(\omega+\nu)\omega}{32\left(\omega+\frac{\nu}{2}\right)}\right]u. \end{aligned}\right\} \qquad (16.24)$$

The roots of the characteristic equation in this case have the form

$$\lambda = \pm\sqrt{\frac{h^2\omega^4}{4\nu^2}-\left[\omega-\frac{\nu}{2}+\frac{h^2(\omega+\nu)\omega}{32\left(\omega+\frac{\nu}{2}\right)}\right]^2}, \qquad (16.25)$$

and, consequently, the zone of instability in second approximation is determined with an accuracy to terms of the second order of smallness inclusive, by the following inequality:

$$2\omega\left[1-\frac{h}{4}-\frac{h^2}{64}\right] < \nu < 2\omega\left[1+\frac{h}{4}-\frac{h^2}{64}\right]. \qquad (16.26)$$

Let us now find the relation between ω and h at which the solution of equation (16.10) will be periodic with the period $\frac{4\pi}{\nu}$. This is possible in the case that a = const in eqs.(16.11) and (16.22). For this, it is necessary that λ, determined by eqs.(16.17) or (16.25), is equal to zero.

Thus the condition which must be satisfied by ω and h, for x to be a periodic

function, will be in first approximation

$$\frac{2\omega}{\nu} = 1 \pm \frac{h}{4}, \tag{16.27}$$

and, in second approximation,

$$\frac{2\omega}{\nu} = 1 \pm \frac{h}{4} + \frac{5h^2}{64}, \tag{16.28}$$

or, with the same degree of accuracy, in first approximation,

$$\frac{4\omega^2}{\nu^2} = 1 \pm \frac{h}{2}, \tag{16.29}$$

and, in second approximation,

$$\frac{4\omega^2}{\nu^2} = 1 \pm \frac{h}{2} + \frac{7h^2}{32}. \tag{16.30}$$

The relations (16.29) and (16.30) constitute the equations of curves in the plane ($\frac{4\omega^2}{\nu^2}$, h) (in first and in second approximation) within which the solution of eq.(16.10) will be periodic, i.e., the equations for the boundary curves of the stability zone of the solutions of eq.(16.10); (Fig.103).

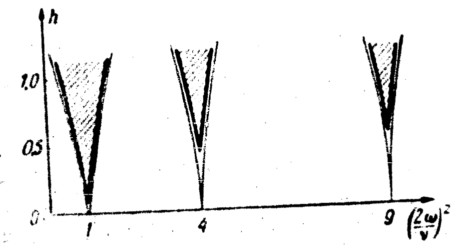

Fig.103

For periodic solutions with a period of $\frac{4\pi}{\nu}$ from eq.(16.11) and (16.22), bearing in mind eq.(16.29) and eq.(16.30), we find the following expressions

$$x_{\rm I} = a_0 \cos\left(\frac{\nu}{2} t + \vartheta_0\right), \tag{16.31}$$

$$x_{\rm II} = a_0 \cos\left(\frac{\nu}{2} t + \vartheta_0\right) - \frac{a_0 h}{16} \cos\left(\frac{3\nu}{2} t + \vartheta_0\right), \tag{16.32}$$

in which the subscript of x indicates the number of the approximation.

Let us now compare these formulas with the results obtained when the periodic

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solutions of eq.(16.10) are directly determined, in the case of small values of h.

For this purpose, let us make use of the method by which Mathieu found the solution of eq.(16.10) and of the equation of the boundary curves, assuming the parameter h to be small. [This method follows directly from Poincaré's method (Bibl.36) of finding periodic solutions.]

Let us find the periodic solution of eq.(16.10) with the period $\frac{4\pi}{\nu}$ in the form of the series

$$x = x_0 + hx_1 + h^2x_2 + \cdots, \tag{16.33}$$

in which x_0, x_1, x_2,... must be periodic functions with the period $\frac{4\pi}{\nu}$.

The expression for ω^2 is also presented in the form of the series

$$\omega^2 = \frac{\nu^2}{4} + h\omega_1 + h^2\omega_2 + \cdots \tag{16.34}$$

On substituting the right sides of eqs.(16.33) and (16.34) in eq.(16.10), and equating the coefficients of equal powers of h, we obtain the following system of equations

$$\left.\begin{aligned}
&\frac{d^2x_0}{dt^2} + \frac{\nu^2}{4}x_0 = 0,\\
&\frac{d^2x_1}{dt^2} + \frac{\nu^2}{4}x_1 = \left(\frac{\nu^2}{4}\cos\nu t - \omega_1\right)x_0,\\
&\frac{d^2x_2}{dt^2} + \frac{\nu^2}{4}x_2 = \left(\frac{\nu^2}{4}\cos\nu t - \omega_1\right)x_1 + (\omega_1\cos\nu t - \omega_2)\,x_0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{16.35}$$

from which we must determine the functions x_0, x_1, x_2,... and the quantities ω_1, ω_2,...

Solving the first equation of the system (16.35), we find

$$x_0 = a_0\cos\left(\frac{\nu}{2}t + \theta_0\right), \tag{16.36}$$

where a_0 and θ_0 are arbitrary constants.

On substituting the value of x_0 from eq.(16.36) in the right side of the second equation of the system (16.35), we have

$$\frac{d^2x_1}{dt^2} + \frac{\nu^2}{4} x_1 = -\omega_1 a_0 \cos\left(\frac{\nu}{2} t + \theta_0\right) + \\ + \frac{\nu^2}{4} a_0 \cos\left(\frac{\nu}{2} t + \theta_0\right) \cos \nu t, \tag{16.37}$$

or

$$\frac{d^2x_1}{dt^2} + \frac{\nu^2}{4} x_1 = -\left(\omega_1 - \frac{\nu^2}{8}\right) a_0 \cos \frac{\nu}{2} t \cos \theta_0 + \\ + \left(\omega_1 + \frac{\nu^2}{8}\right) a_0 \sin \frac{\nu}{2} t \sin \theta_0 + \frac{\nu^2}{8} a_0 \cos\left(\frac{3}{2} \nu t + \theta_0\right). \tag{16.38}$$

For x_1 to be a periodic function with the period $\frac{4\pi}{\nu}$ it is necessary that the coefficients of $\cos \frac{\nu}{2} t$ and $\sin \frac{\nu}{2} t$ in the right-hand side of eq.(16.38) vanish. Then we obtain

$$\omega_1 = \frac{\nu^2}{8}, \quad \sin \theta_0 = 0, \quad \theta = 0$$

or

$$\omega_1 = -\frac{\nu^2}{8}, \quad \cos \theta_0 = 0, \quad \theta_0 = \frac{\pi}{4}.$$

This yields the expression for x_1:

$$x_1 = -\frac{a_0}{16} \cos \frac{3\nu}{2} t + u \cos\left(\frac{\nu}{2} t + \varphi\right), \tag{16.39}$$

or

$$x_1 = \frac{a_0}{16} \sin \frac{3\nu}{2} t + u \cos\left(\frac{\nu}{2} t + \varphi\right). \tag{16.40}$$

On substituting the values of x_0 and x_1 in the right side of the third equation of the system (16.35), we have for the case $\omega_1 = \frac{\nu^2}{8}$, $\theta_0 = 0$:

$$\frac{d^2x_2}{dt^2} + \frac{\nu^2}{4} x_2 = -\left(\frac{\nu^2}{4} \cos \nu t - \omega_1\right) \frac{a}{16} \cos \frac{3\nu}{2} t - \\ - \left(\frac{\nu^2}{4} \cos \nu t - \omega_1\right) u \cos\left(\frac{\nu}{2} t + \varphi\right) + (\omega_1 \cos \nu t - \omega_2) a \cos \frac{\nu}{2} t.$$

By again equating the coefficients of $\cos \frac{\nu}{2} t$ and $\sin \frac{\nu}{2} t$ to zero, we obtain

$$\omega_2 = \frac{\nu^2}{16} - \frac{\nu^2}{128} = \frac{7\nu^2}{128}, \tag{16.41}$$

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$$\frac{\nu^2}{4} u \sin\varphi + \frac{\nu^2}{8} u \sin\varphi = 0, \quad i.e. \quad \sin\varphi = 0, \quad \varphi = 0.$$

By analogy, we obtain for the second case:

$$\left.\begin{aligned} \omega_2 &= \frac{7\nu^2}{128}, \\ \cos\varphi = 0, \quad \varphi &= \frac{\pi}{2}. \end{aligned}\right\} \qquad (16.42)$$

On substituting the resultant values of x_0, x_1, ω_1, ω_2, in the right sides of equations (16.33) and (16.34), we have

$$x = a_0 \cos\frac{\nu}{2} t - \frac{a_0 h}{16} \cos\frac{3\nu}{2} t, \qquad (16.43)$$

$$\omega^2 = \frac{\nu^2}{4} + \frac{h\nu^2}{8} + \frac{7h^2\nu^2}{128}, \qquad (16.44)$$

or

$$x = a_0 \sin\frac{\nu}{2} t + \frac{a_0 h}{16} \sin\frac{3\nu}{2} t, \qquad (16.45)$$

$$\omega^2 = \frac{\nu^2}{4} - \frac{h\nu^2}{8} + \frac{7h^2\nu^2}{128}. \qquad (16.46)$$

In these formulas, for convenience, we have included hu in the total amplitude of the first harmonic a_0.

Putting $\nu = 2$ and $a_0 = 1$, we find

$$\left.\begin{aligned} x &= \cos t - \frac{h}{16}\cos 3t, \\ \omega^2 &= 1 + \frac{h}{2} + \frac{7h^2}{32}; \end{aligned}\right\} \qquad (16.47)$$

$$\left.\begin{aligned} x &= \sin t + \frac{h}{16}\sin 3t, \\ \omega^2 &= 1 - \frac{h}{2} + \frac{7h^2}{32}. \end{aligned}\right\} \qquad (16.48)$$

The expressions (16.47) and (16.48), as was to be expected, coincide with the first two terms in the expansions of the Mathieu functions C_n and S_n (for n = 1) into a Fourier series. In fact, we have (Bibl.53)

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$$\left.\begin{aligned} C_1 &= \cos t - \frac{h}{16}\cos 3t + O(h^2), \\ \omega^2_{C_1} &= 1 + \frac{h}{2} + \frac{7h^2}{32} + O(h^3), \end{aligned}\right\} \qquad (16.49)$$

and likewise

$$\left.\begin{aligned} S_1 &= \sin t + \frac{h}{16}\sin 3t + O(h^2), \\ \omega^2_{S_1} &= 1 - \frac{h}{2} + \frac{7h^2}{32} + O(h^3). \end{aligned}\right\} \qquad (16.50)$$

Let us then discuss the construction of the approximate solutions of eq.(16.10), and let us determine the boundaries of the instability zone in the case $\omega \approx \frac{\nu}{2} p$, where p = 2,3.

For the case of p = 2, i.e., for $\omega \approx \nu$, we use p = 2, q = 2 in eqs.(13.21), (13.23) and (13.26).

Then, for the second approximation, we find the following expression

$$x = a\cos(\nu t + \theta) + \frac{ha\omega^2}{2\nu(\nu + 2\omega)}\cos(2\nu t + \theta) + \frac{h\omega^2 a}{2\nu(\nu - 2\omega)}\cos\theta, \qquad (16.51)$$

where a and θ must be determined from the equation of second approximation:

$$\left.\begin{aligned} \frac{da}{dt} &= + \frac{h^2 a\omega^4}{8\nu^2(2\omega - \nu)}\sin 2\theta, \\ \frac{d\theta}{dt} &= \omega - \nu - \frac{h^2\omega^4}{4\nu(4\omega^2 - \nu^2)} + \frac{h^2\omega^4}{8\nu^2(2\omega - \nu)}\cos 2\theta. \end{aligned}\right\} \qquad (16.52)$$

The system (16.52), as in the preceding case, shows that the condition for the roots of the characteristic equations

$$\left|\omega - \nu - \frac{h^2\omega^4}{4\nu(4\omega^2 - \nu^2)}\right| < \left|\frac{h^2\omega^4}{8\nu^2(2\omega - \nu)}\right| \qquad (16.53)$$

is real so that the instability zone is determined with an accuracy to terms of the second order of smallness, by the inequation

$$4 + \frac{2h^2\omega^4}{\nu^2(4\omega^2 - \nu^2)} - \frac{h^2\omega^4}{\nu^3(2\omega - \nu)} < \left(\frac{2\omega}{\nu}\right)^2 < 4 + \frac{2h^2\omega^4}{\nu^2(4\omega^2 - \nu^2)} + \frac{h^2\omega^4}{\nu^3(2\omega - \nu)} \qquad (16.54)$$

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or, in view of the fact that $\omega \approx \nu$, with the same degree of accuracy, by the inequation

$$4 + \frac{2h^2}{3} - h^2 < \left(\frac{2\omega}{\nu}\right)^2 < 4 + \frac{2h^2}{3} + h^2. \qquad (16.55)$$

For the case of $p = 3$, i.e., for $\omega \approx \frac{3}{2}\nu$, the third approximation must be used since the second approximation gives only a correction refining the value of the natural frequency ω. After several calculations we find

$$\begin{aligned} x = a\cos\left(\frac{3}{2}\nu t + \vartheta\right) - \frac{\omega^2 ha}{2}\left\{\frac{\cos\left(\frac{5}{2}\nu t + \vartheta\right)}{\nu(2\omega+\nu)} - \frac{\cos\left(\frac{1}{2}\nu t + \vartheta\right)}{\nu(2\omega-\nu)}\right\} + \\ + \frac{h^2\omega^4 a}{16\nu^2}\left\{\frac{\cos\left(\frac{7}{2}\nu t + \vartheta\right)}{(2\omega+\nu)(\nu+\omega)} + \frac{\cos\left(\frac{\nu}{2}t - \vartheta\right)}{(2\omega-\nu)(\omega-\nu)}\right\}, \end{aligned} \qquad (16.56)$$

where a and θ must be determined from the system of equations of third approximation

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{h^3\omega^6 a}{3\cdot 2^5\nu^3(2\omega-\nu)(\omega-\nu)}\sin 2\vartheta, \\ \frac{d\vartheta}{dt} &= \omega - \frac{3}{2}\nu - \frac{\omega^4 h^2}{3\cdot 2\nu(4\omega^2-\nu^2)} - \\ &\quad - \frac{h^3\omega^6}{3\cdot 2^5\nu^3(2\omega-\nu)(\omega-\nu)}\cos 2\vartheta. \end{aligned}\right\} \qquad (16.57)$$

It should be mentioned that, in the general case, the expressions for the third approximation are not written out because of their complexity. The structure of eq.(16.57), however, is one more indication that, in concrete cases, even equations of the third approximation can be very simple.

From the system of equations (16.57) the following inequation is obtained for the zone of instability:

$$\begin{aligned} 9 + \frac{2\omega^4h^2}{\nu^2(4\omega^2-\nu^2)} - \frac{\omega^6h^3}{2^5\nu^4(2\omega-\nu)(\omega-\nu)} < \left(\frac{2\omega}{\nu}\right)^2 < 9 + \\ + \frac{2\omega^4h^2}{\nu^2(4\omega^2-\nu^2)} + \frac{\omega^6h^3}{2^5\nu^4(2\omega-\nu)(\omega-\nu)} \end{aligned} \qquad (16.58)$$

or, with the same degree of accuracy,

$$9 + \frac{81h^2}{64} - \frac{3^6h^3}{2^9} < \left(\frac{2\omega}{\nu}\right)^2 < 9 + \frac{81h^2}{64} + \frac{3^6h^3}{2^9}. \qquad (16.59)$$

We also present here the inequation defining the instability zone in the case of $\omega \approx \frac{\nu}{2}$. From eq.(16.20) we have

$$1 - \frac{h}{2} < \left(\frac{2\omega}{\nu}\right)^2 < 1 + \frac{h}{2}. \tag{16.60}$$

An analysis of the inequations (16.55), (16.59), and (16.60), shows that the magnitude (width) of the instability zone decreases with its order of p, as h^p.

Thus the higher resonances $p = 2, 3, \ldots$ may be observed by considering, respectively, the second, third, etc. approximations, while on consideration of the exact solution of eq.(16.10), we obtain an infinite spectrum of resonances.

In Fig.103 we present the first three instability zones constructed according to the inequalities (16.55), (16.59) and (16.60).

We note that, in the presence of damping, i.e., for the equation

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \omega^2(1 - h\cos\nu t)\,x = 0 \tag{16.61}$$

these zones diminish (see the hatched areas in Fig.103). It is not difficult to show that, instead of the inequalities considered, the following expressions are obtained in the presence of friction:

$$1 - \sqrt{\frac{h^2}{4} - \frac{4\delta^2}{\nu^2}} < \left(\frac{2\omega}{\nu}\right)^2 < 1 + \sqrt{\frac{h^2}{4} - \frac{4\delta^2}{\nu^2}}, \tag{16.62}$$

$$4 + \frac{2h^2}{3} - \sqrt{h^4 - 64\frac{\delta^2}{\nu^2}} < \left(\frac{2\omega}{\nu}\right)^2 < 4 + \frac{2h^2}{3} + \sqrt{h^4 - 64\frac{\delta^2}{\nu^2}}, \tag{16.63}$$

$$9 + \frac{81h^3}{64} - \sqrt{\frac{3^{12}h^6}{2^{18}} - 3^4 4\frac{\delta^2}{\nu^2}} < \left(\frac{2\omega}{\nu}\right)^2 < 9 + \frac{81h^2}{64} + \sqrt{\frac{3^{12}h^6}{2^{18}} - 3^4 4\frac{\delta^2}{\nu^2}}. \tag{16.64}$$

The inequations (16.62), (16.63) and (16.64) also contain the additional conditions:

for the first region

$$h > 4\frac{\delta}{\nu}, \tag{16.65}$$

for the second region

$$h > 2\sqrt{2\frac{\delta}{\nu}}, \tag{16.66}$$

for the third region

$$h > \frac{8}{3}\sqrt[6]{\frac{4\delta^2}{9\nu^2}}. \tag{16.67}$$

It is obvious that, in the presence of damping, a far greater depth of modulation of the parameter $h\omega^2$ is required to make the resonance $\omega \approx \nu$ perceptible than in the case of the resonance $\omega \approx \frac{3}{2}\nu$. The resonance $\omega \approx \frac{3}{2}\nu$ is harder to obtain. For this reason the representation of the resonance $\omega \approx \frac{\nu}{2}$ is usually of the greatest practical interest.

Let us now consider parametric excitation in a nonlinear oscillatory system.

We note that the above case shows that, in a linear oscillatory system with parametric variation of mass or rigidity of the system, the position of equilibrium, under certain conditions, becomes unstable. Even at very small values of $\omega^2 h$ (depth of modulation) in the system, oscillations whose amplitude increases without limit arise at a certain frequency ratio.

When dissipative forces are present in a linear system, their influence is manifested only in the conditions under which oscillation is excited. In the presence of dissipation, the depth of modulation at which resonance sets in has a certain lower limit, differing from zero, and depending on the value of the damping decrement. There will be no stationary oscillations when there is friction in a linear system.

The situation is different in a nonlinear oscillatory system. As shown below, resonance occurs whenever the parameters of the oscillatory system under consideration vary by a harmonic law, at a frequency assumed (for the sake of being definite) to be for instance equal or close to double the natural frequency of the system. In this case, stable states of stationary oscillation are possible.

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As an extremely simple example, consider the oscillatory system described by the following differential equation:

$$\frac{d^2x}{dt^2} + \omega^2(1 - h\cos\nu t)\,x + 2\delta\frac{dx}{dt} + \gamma x^3 = 0. \qquad (16.68)$$

Assume that the oscillations described by eq.(16.68) are close to harmonic. Then we will seek the solution of eq.(16.68) that corresponds to the presence of a fundamental submultiple resonance in the system in the form:

$$x = a\cos\left(\frac{\nu}{2}t + \theta\right), \qquad (16.69)$$

where, according to eq.(13.25), a and θ must satisfy the following system of equations:

$$\left.\begin{aligned} \frac{da}{dt} &= -\delta a - \frac{ah\omega^2}{2\nu}\sin 2\theta, \\ \frac{d\theta}{dt} &= \omega - \frac{\nu}{2} + \frac{3\gamma a^2}{4\nu} - \frac{h\omega^2}{2\nu}\cos 2\theta. \end{aligned}\right\} \qquad (16.70)$$

To obtain the stationary value of the amplitude and phase of the oscillations, let us equate the right sides of the system (16.70) to zero.

This gives the relations:

$$\left.\begin{aligned} -\delta a - \frac{ah\omega^2}{2\nu}\sin 2\theta &= 0, \\ \omega - \frac{\nu}{2} + \frac{3\gamma a^2}{4\nu} - \frac{h\omega^2}{2\nu}\cos 2\theta &= 0. \end{aligned}\right\} \qquad (16.71)$$

Eliminating from them the phase θ, we can find, with an accuracy to terms of the first order of smallness inclusive, the following relation between the amplitude a and the frequency of modulation ν:

$$a^2 = \frac{4}{3\gamma}\left[\left(\frac{\nu}{2}\right)^2 - \omega^2 \mp \frac{1}{2}\sqrt{h^2\omega^4 - 4\nu^2\delta^2}\,\right]. \qquad (16.72)$$

Using this relation, we can construct the resonance curve.

In the case where $\gamma > 0$, we obtain the resonance curve shown in Fig.104. On analyzing this curve we see that as ν increases, beginning with small values, the oscillation in this system will be absent until ν reaches values corresponding to the point A. When ν reaches the point A, oscillations will arise in the system, and

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when ν increases further, the amplitude of these oscillations will vary along the upper branch AB of the resonance curve. At the point B, the oscillations lose their stability and decay.

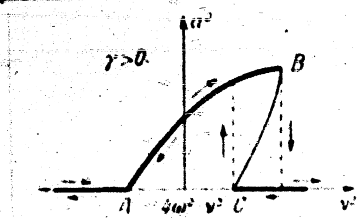

Fig. 104

At ν decreasing from large values, the oscillations are abruptly excited at the point C (hard excitation), and as ν then decreases, the amplitude of the oscillations will vary along the curve AB.

In the case where $\gamma < 0$, we obtain an analogous picture, except that the resonance curve will be inclined toward the small values of ν (Fig. 105).

To determine the boundaries of the zone of synchronization, the right side of the expression for a must be equated to zero.

In first approximation, the zone of resonance will be

$$\omega^2 - \frac{1}{2}\sqrt{h^2\omega^4 - 16\omega^2\delta^2} < \left(\frac{\nu}{2}\right)^2 < \omega^2 + \frac{1}{2}\sqrt{h^2\omega^4 - 16\omega^2\delta^2}, \qquad (16.73)$$

so that the width of the resonance zone will become

$$\Delta = \sqrt{h^2\omega^4 - 16\omega^2\delta^2}. \qquad (16.74)$$

We note that the presence of damping reduces the interval AC within which parametric resonance takes place.

It is obvious that Δ will be real if the inequation

$$h > \frac{4\delta}{\omega}, \qquad (16.75)$$

is satisfied. This inequation, as stated above, determines the minimum depth of modulation necessary for parametric resonance at a given damping.

Let us consider still another case of parametric resonance in an oscillatory system with nonlinear friction.

In the case of parametric excitation of a circuit with a vacuum tube (Fig. 23) the equation of oscillation will be

$$\frac{d^2x}{dt^2} + 2(\lambda_0 + \lambda_2 x^2)\frac{dx}{dt} + \omega^2(1 - h\cos\nu t)x = 0. \qquad (16.76)$$

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Assume that in the absence of parametric excitation, i.e., at h = 0, the system is not self-excited. For this, it is necessary that $\lambda_0 > 0$.

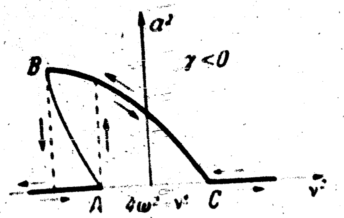

Fig. 105

Let us set up the equation of first approximation. We have the following:

$$\left.\begin{aligned} \frac{da}{dt} &= -\lambda_0 a - \frac{\lambda_2 a^3}{4} - \frac{ah\omega^2}{2\nu}\sin 2\theta, \\ \frac{d\theta}{dt} &= \omega - \frac{\nu}{2} - \frac{h\omega^2}{2\nu}\cos 2\theta. \end{aligned}\right\}$$

To determine the stationary values of a and θ we equate the right sides of eq.(16.77) to zero:

$$\left.\begin{aligned} \lambda_0 a + \frac{\lambda_2 a^3}{4} + \frac{ah\omega^2}{2\nu}\sin 2\theta &= 0, \\ \omega - \frac{\nu}{2} - \frac{h\omega^2}{2\nu}\cos 2\theta &= 0. \end{aligned}\right\} \tag{16.78}$$

By eliminating θ from the relation so obtained, we find, with the degree of accuracy adopted by us, the following relation between the amplitude of the oscillation a and the frequency variation of the parameter ν:

$$a^2 = \frac{2}{\lambda_2\nu}\sqrt{h^2\omega^4 - 4\left(\omega^2 - \left(\frac{\nu}{2}\right)^2\right)^2} - 4\frac{\lambda_0}{\lambda_2}. \tag{16.79}$$

This relation is then used for constructing the resonance curve (Fig. 106), and also for finding the conditions of parametric excitation, the maximum amplitude of the excitation, the boundaries of the resonant region, etc.

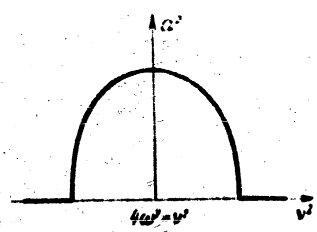

Fig. 106

Section 17. Action of Periodic Forces on a Relaxation System

Let us now investigate the influence of an external perturbation on a relaxation oscillatory system characterized by an equation of the type

$$\frac{dx}{dt} = \Phi(x) + \varepsilon E\cos\nu t, \tag{17.1}$$

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where, $\Phi(x)$, as in Chapter I, represents a certain two-valued function over the interval (a,b).

For constructing the solution of eq.(17.1) it is expedient first to transform eq.(17.1) in order to eliminate the multi-valued function $\Phi(x)$ from it. For this, we will start from a certain partial periodic solution of the equation of free relaxation oscillations:

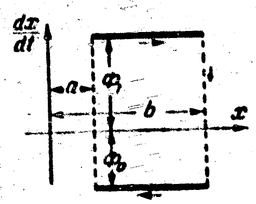

Fig. 107

$$\frac{dx}{dt}=\Phi(x). \qquad (17.2)$$

To be more definite, let us take the solution of eq.(17.2) in which the value x assumes its minimum value at $t = 0$.

Denoting the frequency of the free relaxation oscillations by ω, we write this periodic solution in the form

$$x=z(\omega t), \qquad (17.3)$$

where $z(\varphi)$ is a certain periodic function of φ with a period of 2π.

Taking into consideration the results of Section 10, it becomes obvious that the derivative $z'(\varphi)$ during a single period undergoes a discontinuity twice and that, in absolute value, it is always greater than some positive constant.

For instance, if the two-valued function $\Phi(x)$ has the following values for the upper and lower branch, respectively:

$$\left.\begin{array}{ll}\Phi(x)=\Phi_1=\text{const}, & a<x<b,\\ \Phi(x)=-\Phi_0=\text{const}, & a<x<b,\end{array}\right\} \qquad (17.4)$$

then the solution of eq.(17.2) may be represented in the form (Fig.107)

$$x=a+\Phi_1 t, \quad 0<t<\frac{b-a}{\Phi_1},$$

$$x=b+\Phi_0\left[\frac{b-a}{\Phi_1}-t\right], \quad \frac{b-a}{\Phi_1}<t<(b-a)\frac{\Phi_0+\Phi_1}{\Phi_0\Phi_1}, \qquad (17.5)$$

$$T=\frac{2\pi}{\omega}=(b-a)\frac{\Phi_0+\Phi_1}{\Phi_0\Phi_1} \qquad (17.6)$$

and, consequently,

$$\omega=\frac{2\pi}{b-a}\frac{\Phi_0\Phi_1}{\Phi_0+\Phi_1}. \qquad (17.7)$$

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Let us denote:

$$\varphi_0 = 2\pi \frac{\Phi_0}{\Phi_0 + \Phi_1},$$

then the periodic solution $z(\varphi)$ may be written as follows:

$$\left.\begin{aligned} z(\varphi) &= a + \frac{b-a}{\varphi_0}\varphi, \quad 0 < \varphi < \varphi_0, \\ z(\varphi) &= a + \frac{b-a}{\varphi_0}\frac{\Phi_0}{\Phi_1}(2\pi - \varphi), \quad \varphi_0 < \varphi < 2\pi. \end{aligned}\right\} \qquad (17.8)$$

We note that, since eq.(17.8) represents a solution of eq.(17.2), the function $z(\varphi)$ must identically satisfy the following relation:

$$\omega z'(\varphi) = \Phi[z(\varphi)]. \qquad (17.9)$$

We now perform a substitution of variables in the equation describing the forced oscillations, eq.(17.1).

We replace the unknown x by a new unknown φ, using the formula

$$x = z(\varphi). \qquad (17.10)$$

On differentiating eq.(17.10) and substituting in eq.(17.1), we obtain

$$z'(\varphi)\frac{d\varphi}{dt} = \Phi[z(\varphi)] + \varepsilon E \cos \nu t, \qquad (17.11)$$

or, in view of the identity (17.9):

$$\frac{d\varphi}{dt} = \omega + \frac{\varepsilon E \cos \nu t}{z'(\varphi)}. \qquad (17.12)$$

The transformed equation (17.12) no longer contains multi-valued functions in the right side.

For convenience in constructing the approximate solutions of differential equations, it is usually desirable that the right side be a regular function. In equation (17.12), the right side, in view of the presence of the discontinuous function $z'(\varphi)$ in the denominator, does not satisfy the condition of regularity.

In order to regularize eq.(17.2), it is sufficient to interchange the roles of the variables t and φ and subsequently to consider φ the independent variable, and t an unknown function of φ, determined by the differential equation

$$\frac{d(\nu t)}{d\varphi} = \frac{\nu}{\omega + \dfrac{\varepsilon E \cos \nu t}{z'(\varphi)}}. \qquad (17.13)$$

If we denote by γ a positive constant, such that

$$|\Phi(x)| > \gamma, \quad a < x < b, \tag{17.14}$$

then eqs.(17.9) and (17.10) will yield

$$\omega\,|z'(\varphi)| > \gamma. \tag{17.15}$$

Assume that the amplitude εE of the external disturbing force is less than γ. Then the denominator in the right side of eq.(17.13) is positive, and the right side of eq.(17.13) is itself an analytic function of the unknown t. Equations of the type of eq.(17.13) have been investigated by Poincaré and Denjoy. Their results, however, clarify only the qualitative character of the solutions. To obtain a technique allowing quantitative calculations, let us make use of the method of the mean which has been briefly described in Chapter I.

In order to apply the results of Section 1 to eq.(17.13), let us expand the right side of eq.(17.13) into a power series of ε. We have

$$\frac{d(\nu t)}{d\varphi} = \frac{\nu}{\omega} - \frac{\varepsilon\nu E}{\omega^2}\frac{\cos\nu t}{z'(\varphi)} + \frac{\varepsilon^2\nu E^2}{\omega^3}\frac{\cos^2\nu t}{[z'(\varphi)]^3} - \varepsilon^3\ldots \tag{17.16}$$

Let us investigate eq.(17.16) for the resonant case. Assume that the ratio $\frac{\nu}{\omega}$ is close to some rational number $\frac{p}{q}$ where, as before, p and q, generally speaking, are small prime numbers.

Then, putting

$$\frac{\nu}{\omega} = \frac{p}{q} + \varepsilon\Delta \tag{17.17}$$

and introducing the new variable τ from the formula

$$\tau = \nu t - \frac{p}{q}\varphi,$$

eq.(17.16) can finally be written in the form

$$\frac{d\tau}{d\varphi} = \varepsilon\left\{\Delta - \frac{p}{q}\frac{E}{\omega}\frac{\cos\left(\tau + \frac{p}{q}\varphi\right)}{z'(\varphi)}\right\} + \varepsilon^2\left\{\frac{pE^2}{q\omega^2}\frac{\cos^2\left(\tau + \frac{p}{q}\varphi\right)}{[z'(\varphi)]^3} - \frac{\Delta E}{\omega}\frac{\cos\left(\tau + \frac{p}{q}\varphi\right)}{z'(\varphi)}\right\} + \varepsilon^3\ldots \tag{17.18}$$

We have agreed above to denote an equation of the type of (17.18) as an equation of the standard form.

The approximate solution of this equation may be constructed on the basis of the principle of averaging.

In first approximation, according to the results of Section 1, Chapter I, the solution of eq. (17.18) will be

$$\tau = \xi, \tag{17.19}$$

where ξ is determined from the mean equation

$$\frac{d\xi}{d\varphi} = \varepsilon\Delta - \frac{\varepsilon pE}{q\omega} \underset{\varphi}{M} \left\{ \frac{\cos\left(\xi + \frac{p}{q}\varphi\right)}{z'(\varphi)} \right\}. \tag{17.20}$$

Let us further discuss the operation of taking the mean in the right side of the resultant equation. For this, it is necessary to expand the function $\frac{1}{\omega z'(\varphi)}$ into a Fourier series.

We have

$$\frac{1}{\omega z'(\varphi)} = \frac{1}{\Phi[z(\varphi)]} = A_0 + \sum_{n \neq 0} A_n \cos(n\varphi + \vartheta_n). \tag{17.21}$$

We note now that the expression

$$\underset{\varphi}{M} \left\{ \cos(n\varphi + \vartheta_n) \cos\left(\xi + \frac{p}{q}\varphi\right) \right\}$$

can be different from zero only where $\frac{p}{q} = n$. In this case we have:

$$\underset{\varphi}{M} \{\cos(n\varphi + \vartheta_n) \cos(\xi + n\varphi)\} = \frac{1}{2}\cos(\xi - \vartheta_n).$$

Thus, if $\frac{p}{q} \neq n$, where n is an integer, the equation of first approximation (17.20) degenerates into the following:

$$\frac{d\xi}{d\varphi} = \varepsilon\Delta, \tag{17.22}$$

from which we find

$$\tau = \xi = \varepsilon\Delta\varphi + \text{const},$$

i.e.,

$$\nu t - \frac{p}{q}\varphi = \left(\frac{\nu}{\omega} - \frac{p}{q}\right)\varphi + \text{const}$$

or

$$\varphi = \omega t + \varphi_0. \tag{17.23}$$

Consequently, in first approximation, we obtain

$$x = z(\omega t + \varphi_0). \qquad (17.24)$$

Thus, in the case when $\frac{p}{q} \neq n$, we obtain for forced oscillations in first approximation the same kind of expression as for free oscillations when the external force $\varepsilon E \cos \nu t$ does not act on the system.

Thus, in first approximation, the influence of a small external force on the form and frequency of the oscillations is negligible unless its frequency is sufficiently close to one of the overtones of the natural frequency.

Consider now the case when $\frac{p}{q}$ is equal to a certain integer m, which corresponds to the subharmonic resonance $\omega \approx \frac{\nu}{m}$.

From eq.(17.20) we find

$$\frac{d\xi}{d\varphi} = \frac{\nu}{\omega} - m - \frac{\varepsilon m A_m}{2} \cos(\xi - \vartheta_m). \qquad (17.25)$$

The equation so obtained is considerably simpler than the corresponding equation of first approximation (13.40) for the system considered in Sect.13 of the present Chapter, where we obtained a system of two differential equations in two unknowns, namely the amplitude and full phase of the oscillation. In the case under discussion (that of relaxation oscillations) we have only one differential equation with respect to the phase angle ξ, which, in addition, is integrated by mechanical quadrature.

The character of the solution in eq.(17.25) may be found directly, without first integrating it.

Let, for instance,

$$\left| \frac{\nu}{\omega} - m \right| < \left| \frac{\varepsilon m A_m}{2} \right|. \qquad (17.26)$$

Then the derivative $\frac{d\xi}{d\varphi}$ will be an alternating function of ξ of the type shown in Fig. 108.

Thus it is obvious that there exist constant solutions ξ_i, which are roots of the equation

$$F(\xi) = \frac{\nu}{\omega} - m - \frac{\varepsilon m A_m}{2} \cos(\xi - \vartheta_m) = 0. \qquad (17.27)$$

In this case, the solutions for which

$$F'(\xi) = \frac{\varepsilon m A_m}{2} \sin(\xi - \vartheta_m) > 0,$$

are unstable, while the solutions for which

$$F'(\xi) = \frac{\varepsilon m A_m}{2} \sin(\xi - \vartheta_m) < 0,$$

are stable.

Since, in the case under consideration, $\xi = \tau = \nu t - m\varphi$, we have

$$x = z\left[\frac{\nu t - \xi}{m}\right], \tag{17.28}$$

whence it is obvious that forced relaxation oscillations, with the passage of time, approach steady periodic oscillations, corresponding to the different roots of equa-

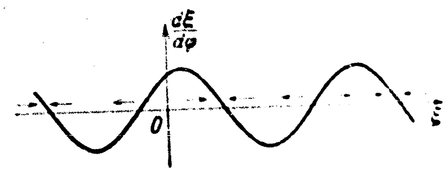

Fig. 108

tion (17.27), and occurring at a frequency exactly equal to the subharmonic $\frac{\nu}{m}$ of the frequency ν of the external force.

Thus, for values of the frequency ν lying inside the resonance band determined by the inequation (17.26)

$$m - \left|\frac{\varepsilon m A_m}{2}\right| < \frac{\nu}{\omega} < m + \left|\frac{\varepsilon m A_m}{2}\right|, \tag{17.29}$$

the phenomenon of synchronization occurs.

The width of the resonant zone, in first approximation,

$$|\varepsilon m A_m|,$$

is obviously proportional to the amplitude of the external force.

Consider now the case when ν lies outside the resonant zone, so that

$$\left|\frac{\nu}{\omega}-m\right| > \left|\frac{\varepsilon m A_m}{2}\right|. \qquad (17.30)$$

In this case, eq.(17.25), indicates clearly that the derivative $\frac{d\xi}{d\varphi}$ has a constant sign which is the same as the sign of the difference $\frac{\nu}{\omega} - m$.

On integrating eq.(17.25), we get

$$\varphi = \int_0^{\xi} \frac{d\xi}{\frac{\nu}{\omega} - m - \frac{\varepsilon m A_m}{2}\cos(\xi - \vartheta_m)} + \text{const},$$

whence we find

$$\varphi = \frac{\xi}{\alpha} + \frac{1}{\alpha} f(\xi) + \vartheta_0, \qquad (17.31)$$

where

$$\alpha = \left(\frac{\nu}{\omega} - m\right)\sqrt{1 - \frac{\varepsilon^2 m^2 A_m^2}{4\left(m - \frac{\nu}{\omega}\right)^2}}, \qquad (17.32)$$

in which θ_o is an arbitrary constant, $f(\xi)$ is a periodic function of ξ with the period of 2π:

$$f(\xi) = \frac{1}{\pi}\operatorname{arctg}\frac{\cos\left(\frac{\xi - \vartheta_m - \beta}{2}\right)}{\cos\left(\frac{\xi - \vartheta_m + \beta}{2}\right)}. \qquad (17.33)$$

Here

$$\beta = \arccos\frac{\varepsilon m A_m}{2\left(m - \frac{\nu}{\omega}\right)}, \quad 0 < \beta < \pi.$$

On transforming eq.(17.31), we get

$$\xi = \alpha(\varphi - \vartheta_0) + F\{\alpha(\varphi - \vartheta_0)\}, \qquad (17.34)$$

where $F(\theta)$ is a periodic function of θ with the period 2π.

Noting that, in the approximation adopted,

$$\xi = \tau = \nu t - m\varphi,$$

we obtain from eq.(17.34):

$$\nu t = m\varphi + \alpha(\varphi - \vartheta_0) + F\{\alpha(\varphi - \vartheta_0)\}. \tag{17.35}$$

Assume that

$$\alpha(\varphi - \vartheta_0) = \theta;$$

Then,

$$\varphi = \vartheta_0 + \frac{\theta}{\alpha} \tag{17.36}$$

and

$$(m + \alpha)\theta + \alpha F(\theta) = \alpha(\nu t - m\vartheta_0),$$

whence, solving this equation with respect to θ, we find:

$$\theta = \frac{\alpha(\nu t - m\vartheta_0)}{m + \alpha} + \alpha\sigma\left\{\frac{\alpha(\nu t - m\vartheta_0)}{m + \alpha}\right\}, \tag{17.37}$$

where $\sigma(\theta)$ is a periodic function of θ with the period 2π.

On substituting the value of θ from eq.(17.37) in the right side of eq.(17.31)

$$\varphi = \Omega_p t + \varphi_1 + \sigma[m(\Omega_s - \Omega_p)t - m\varphi_1], \tag{17.38}$$

where the following notation is used:

$$\Omega_p = \frac{\nu}{m + \alpha}, \quad \Omega_s = \frac{\nu}{m},$$

$$\varphi_1 = \frac{\vartheta_0 \alpha}{m + \alpha} = \text{const}.$$

On substituting eq.(17.38) in eq.(17.28), we find the final approximate expression for the forced relaxation oscillations in the form

$$x = z\{\Omega_p t + \varphi_1 + \sigma[m(\Omega_s - \Omega_p)t - m\varphi_1]\}. \tag{17.39}$$

Thus, in the case under consideration, the oscillations are multiperiodic and occur with two fundamental frequencies: with the frequency Ω_p which can be called the variable natural frequency, since $\Omega_p \to \omega$ for $\varepsilon \to 0$, and with the beat frequency

$$|m(\Omega_s - \Omega_p)| = \left|\frac{\alpha\nu}{m + \alpha}\right| \approx \sqrt{(\nu - m\omega)^2 - \frac{\varepsilon^2 m^2 A_m^2}{4}}, \tag{17.40}$$

representing the difference tone between the frequency of the external force $\nu = m\Omega_s$ and the m^{th} overtone of the variable natural frequency.

We note that, as ν approaches the boundary of the resonance zone, $a \to 0$, and therefore the beat frequency also tends toward zero.

In addition, it is not difficult to show that, on departure from resonance, the intensity of the beats, determined by the function σ, diminishes so that the variable natural frequency Ω_p approaches its value ω corresponding to free oscillations.

Next, we construct the second approximation: For this purpose, let us first find the expression for the refined first approximation.

Let us consider first the general case of an arbitrary rational value for the ratio $\frac{p}{q}$.

Making use of the Fourier expansion of eq. (17.21), we obtain the following expression for the coefficient of the first power of ε in the right side of eq. (17.20):

$$\Delta - \frac{p}{q\omega} \frac{\cos\left(\tau + \frac{p}{q}\varphi\right)}{z'(\varphi)} = \Delta - \frac{p}{q} A_0 \cos\left(\tau + \frac{p}{q}\varphi\right) - \\ - \frac{p}{q} \sum_{n \neq 0} \frac{A_n}{2} \left\{ \cos\left[\left(n + \frac{p}{q}\right)\varphi + \tau + \vartheta_n\right] + \right. \\ \left. + \cos\left[\left(n - \frac{p}{q}\right)\varphi - \tau + \vartheta_n\right] \right\}.$$

The refined first approximation will, therefore, have the following form:

$$\tau = \xi + \varepsilon u(\varphi, \xi), \tag{17.41}$$

where

$$u(\varphi, \xi) = -A_0 \sin\left(\xi + \frac{p}{q}\varphi\right) - \frac{p}{q} \sum_{n \neq 0} \frac{A_n}{2} \frac{\sin\left\{\left(n + \frac{p}{q}\right)\varphi + \xi + \vartheta_n\right\}}{n + \frac{p}{q}} - \\ - \frac{p}{q} \sum_{\left(\substack{n \neq \frac{p}{q} \\ n \neq 0}\right)} \frac{A_n}{2} \frac{\sin\left\{\left(n - \frac{p}{q}\right)\varphi - \xi + \vartheta_n\right\}}{n - \frac{p}{q}}. \tag{17.42}$$

On substituting the value of τ from eq. (17.41) in eq. (17.20) and averaging over φ, we obtain the equation of second approximation:

$$\frac{d\xi}{d\varphi} = \varepsilon\left\{\Delta - \frac{p}{q} \underset{\varphi}{M}\left[\frac{\cos\left(\xi+\frac{p}{q}\varphi\right)}{\omega z'(\varphi)}\right]\right\} + \varepsilon^2 \underset{\varphi}{M}\left[\frac{p}{q}\,\frac{\sin\left(\xi+\frac{p}{q}\varphi\right)}{\omega z'(\varphi)}\, u(\varphi,\xi) + \frac{p}{q}\,\frac{\cos^2\left(\xi+\frac{p}{q}\varphi\right)}{[\omega z'(\varphi)]^2} - \frac{\Delta}{\omega}\,\frac{\cos\left(\xi+\frac{p}{q}\varphi\right)}{z'(\varphi)}\right]. \qquad (17.43)$$

Consider first the nonresonant case when the ratio $\frac{p}{q}$ is not equal either to a whole number or to half a whole number, and when, consequently, the frequency of the external force does not lie close to the overtones of the natural frequency, $n\omega$ nor to half of it, $\frac{n\omega}{2}$.

We note that in the adopted nonresonant case the following inequation are obtained for any integers n and m:

$$n \neq \frac{p}{q}, \quad n + \frac{p}{q} \neq m - \frac{p}{q}.$$

Further, eq.(17.21) yields

$$\frac{\cos\left(\xi+\frac{p}{q}\varphi\right)}{\omega z'(\varphi)} = A_0\cos\left(\xi+\frac{p}{q}\varphi\right) + \frac{1}{2}\sum_{n\neq 0} A_n\left\{\cos\left[\left(n+\frac{p}{q}\right)\varphi+\xi+\vartheta_n\right] + \cos\left[\left(n-\frac{p}{q}\right)\varphi-\xi+\vartheta_n\right]\right\}, \qquad (17.44)$$

$$\frac{\sin\left(\xi+\frac{p}{q}\varphi\right)}{\omega z'(\varphi)} = A_0\sin\left(\xi+\frac{p}{q}\varphi\right) + \frac{1}{2}\sum_{n\neq 0} A_n\left\{\sin\left[\left(n+\frac{p}{q}\right)\varphi+\xi+\vartheta_n\right] - \sin\left[\left(n-\frac{p}{q}\right)\varphi-\xi+\vartheta_n\right]\right\}.$$

For this reason, taking account of eqs.(17.42) and (17.44), we may write

$$\underset{\varphi}{M}\left\{\frac{\cos\left(\xi+\frac{p}{q}\varphi\right)}{\omega z'(\varphi)}\right\} = 0, \quad \underset{\varphi}{M}\left\{\frac{\cos^2\left(\xi+\frac{p}{q}\varphi\right)}{[\omega z'(\varphi)]^2}\right\} = \frac{1}{2}A_0^2 + \frac{1}{4}\sum_{n\neq 0} A_n^2, \; -$$

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$$\underset{\varphi}{M}\left\{\frac{\sin\left(\xi+\frac{p}{q}\varphi\right)}{\omega z'(\varphi)}u(\varphi,\xi)\right\} = -\frac{A_0^2}{2}+\frac{p}{q}\sum_{n\neq 0}\frac{A_n^2}{8}\left[\frac{1}{n-\frac{p}{q}}-\frac{1}{n+\frac{p}{q}}\right].$$

The equation of second approximation (17.43) then takes the following form:

$$\frac{d\xi}{d\varphi} = \varepsilon\Delta+\varepsilon^2\gamma = \frac{\nu}{\omega}-\frac{p}{q}+\varepsilon^2\gamma, \tag{17.45}$$

where

$$\gamma = \frac{p}{q}\sum_{n\neq 0}\frac{A_n^2}{4}\,\frac{n^2}{n^2-\frac{p^2}{q^2}}. \tag{17.46}$$

Integrating eq.(17.45), we find

$$\xi = \left[\frac{\nu}{\omega}-\frac{p}{q}+\varepsilon^2\gamma\right]\varphi+\xi_0,$$

where ξ_0 is an arbitrary constant. Bearing in mind eq.(17.41), we obtain the following formulas of second approximation

$$\tau = \nu t-\frac{p}{q}\varphi = \left[\frac{\nu}{\omega}-\frac{p}{q}+\varepsilon^2\gamma\right]\varphi+\xi_0+ \\ +\varepsilon u\left[\varphi,\left(\frac{\nu}{\omega}-\frac{p}{q}+\varepsilon^2\gamma\right)\varphi+\xi_0\right],$$

from which, with an accuracy to terms of the second order of smallness inclusive, we obtain

$$\varphi = \frac{\nu t-\xi_0}{\frac{\nu}{\omega}+\varepsilon^2\gamma} - \\ -\frac{\omega}{\nu}\varepsilon u\left\{\frac{\nu t-\xi_0}{\frac{\nu}{\omega}+\varepsilon^2\gamma},\ \left[\frac{\nu}{\omega}-\frac{p}{q}+\varepsilon^2\gamma\right]\frac{\nu t-\xi_0}{\frac{\nu}{\omega}+\varepsilon^2\gamma}+\xi_0\right\}. \tag{17.47}$$

Now, putting

$$\Omega_p = \frac{\omega}{1+\varepsilon^2\gamma\frac{\omega}{\nu}}, \qquad \varphi_0 = -\Omega_p\frac{\xi_0}{\nu}, \tag{17.48}$$

we have

$$\varphi = \Omega_p t+\varphi_0-\varepsilon u\left\{\Omega_p t+\varphi_0,\ \left(\nu-\frac{p}{q}\Omega_p\right)t-\frac{p}{q}\varphi_0\right\}\frac{\omega}{\nu}. \tag{17.49}$$

Thus, in the nonresonant case we obtain the following expressions of the second

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approximation for forced relaxation oscillations:

$$x = z(\varphi),$$

$$\varphi = \Omega_p t + \varphi_0 + \\ + \frac{p\omega}{q\nu} \sum_{n \neq 0} \frac{\varepsilon A_n}{2\left(n + \frac{p}{q}\right)} \sin\left[n(\Omega_p t + \varphi_0) + \nu t + \theta_n\right] + \\ + \frac{p\omega}{q\nu} \sum_{n \neq 0} \frac{\varepsilon A_n}{2\left(n - \frac{p}{q}\right)} \sin\left[n(\Omega_p t + \varphi_0) - \nu t + \theta_n\right] + \varepsilon A_0 \sin \nu t, \tag{17.50}$$

where, according to eq. (17.48), with the degree of accuracy adopted by us, it follows that

$$\Omega_p = \omega - \sum_{n \neq 0} \frac{\varepsilon^2 A_n^2}{4} \frac{n^2\omega}{n^2 - \frac{p^2}{q^2}}. \tag{17.51}$$

From the formulas of second approximation so obtained it is not difficult to eliminate the auxiliary quantity, the ratio $\frac{p}{q}$. Since the difference $\frac{\nu}{\omega} - \frac{p}{q}$ is of the first order of smallness, eq. (17.50) is valid with an accuracy to terms of the second order of smallness, while eq. (17.51) holds to quantities of the third order of smallness. With the same degree of accuracy, we may write

$$\left.\begin{aligned} & x = z(\varphi), \\ & \varphi = \Omega_p t + \varphi_0 + \sum_{n \neq 0} \frac{\varepsilon A_n \omega}{2(n\omega + \nu)} \sin\{n(\Omega_p t + \varphi_0) + \nu t + \theta_n\} + \\ & \quad + \sum_{n \neq 0} \frac{\varepsilon A_n \omega}{2(n\omega - \nu)} \sin\{n(\Omega_p t + \varphi) - \nu t + \theta_n\} + \varepsilon A_0 \sin \nu t, \\ & \Omega_p = \omega - \sum_{n \neq 0} \frac{\varepsilon^2 A_n^2}{8} \left[\frac{n\omega}{n\omega - \nu} + \frac{n\omega}{n\omega + \nu}\right]. \end{aligned}\right\} \tag{17.52}$$

The solution thus found corresponds to the asynchronous state of oscillations. Here the oscillations will be quasi-periodic with two fundamental frequencies ν and Ω_p.

The variation in the phase angle φ is represented here as a rotation at constant angular velocity equal to Ω_p, on which oscillations of small amplitude, with frequencies ν, $n\Omega_p - \nu$, $n\Omega_p + \nu$ are superimposed.

Let us now construct the second approximation for the resonant case.

~~For investigating the resonant case we must take the ratio~~ $\frac{p}{q}$ ~~as equal to an integer, or to a half-integer.~~ If we put $\frac{p}{q} = m$, the resultant equation of second ap-

proximation will differ from equation (17.25) by terms of the second order of smallness. With this equation, we can refine the position and width of the resonant zone, refine the value of the variable natural frequency of the asynchronous oscillations, etc.

Without dwelling on this, let us consider the case when the ratio $\frac{p}{q}$ is a half-integer:

$$\frac{p}{q} = \frac{2m+1}{2}.$$

To elucidate the operation of averaging in the equation of second approximation (17.23), we note that eq.(17.42) gives

$$\begin{aligned} u(\varphi, \xi) = & -A_0 \sin\left(\xi + \frac{2m+1}{2}\varphi\right) - \\ & -\frac{2m+1}{2} \sum_{n \neq 0} \frac{A_n}{2} \frac{\sin\left[\left(n + \frac{2m+1}{2}\right)\varphi + \xi + \vartheta_n\right]}{n + \frac{2m+1}{2}} - \\ & -\frac{2m+1}{2} \sum_{n \neq 0} \frac{A_n}{2} \frac{\sin\left[\left(n - \frac{2m+1}{2}\right)\varphi - \xi + \vartheta_n\right]}{n - \frac{2m+1}{2}}; \end{aligned} \tag{17.53}$$

On substituting the value of eq.(17.53) in the right side of eq.(17.43), after a number of calculations, we find

$$\frac{d\xi}{dt} = \frac{\nu}{\omega} - \frac{2m+1}{2} + \varepsilon^2\gamma - \varepsilon^2 \frac{2m+1}{4} S_m \cos(2\xi + \psi_m), \tag{17.54}$$

where the following notation is used:

$$\left.\begin{aligned} S_m \sin\psi_m &= \underset{\varphi}{M}\left\{\frac{\sin(2m+1)\varphi}{[\omega z'(\varphi)]^2}\right\}, \\ S_m \cos\psi_m &= -\underset{\varphi}{M}\left\{\frac{\cos(2m+1)\varphi}{[\omega z'(\varphi)]^2}\right\}. \end{aligned}\right\} \tag{17.55}$$

Equation (17.54) so obtained differs from eq.(17.45) for the nonresonant case by the presence of the summand

$$\varepsilon^2 \frac{2m+1}{4} S_m \cos(2\xi - \psi_m).$$

As in the case of the first approximation, it is obvious that the resonant zone is defined by the inequation

$$-\varepsilon^2 \frac{2m+1}{4} S_m < \frac{\nu}{\omega} - \frac{2m+1}{2} + \varepsilon^2\gamma < \varepsilon^2 \frac{2m+1}{4} S_m, \tag{17.56}$$

or, when introducing the variable natural frequency Ω_p, this zone is defined with the same degree of accuracy by the inequation

$$-\varepsilon^2 \frac{2m+1}{4} S_m < \frac{\nu}{\Omega_p} - \frac{2m+1}{2} < \varepsilon^2 \frac{2m+1}{4} S_m. \qquad (17.57)$$

Thus, in considering this first approximation, we have found the resonant zones only for values of ν lying in the neighborhood of $m\omega$, their width being proportional to the first power of ε. In the second approximation we discover additional resonant zones for values of ν lying in the neighborhood of $\frac{2m+1}{2}\omega$, where the width of these "secondary" zones is proportional to the square of ε.

An analysis of the higher approximations would likewise indicate the presence of resonant zones for $\nu \approx \frac{p}{q}\omega$, $q = 3, 4, \ldots$ with a width of the order of ε^q.

Section 18. Nonlinear Systems with Slowly Varying Parameters

In the preceding Sections we have considered nonlinear oscillatory systems in which the time t entered under the sign of the trigonometric functions (external periodic influence).

In many cases, however, the equations describing an oscillatory process may contain several coefficients that also depend on time. In the case where these coefficients vary "slowly" with time ("slowly" with respect to the natural unit of time, the period of natural oscillations), we arrive at a consideration of the following nonlinear differential equation with slowly varying coefficients:

$$\frac{d}{dt}\left[m(\tau)\frac{dx}{dt}\right] + k(\tau)x = \varepsilon F\left(\tau, \theta, x, \frac{dx}{dt}\right), \qquad (18.1)$$

in which, as before, ε is a small positive parameter; $\tau = \varepsilon t$ is the "slow" time, $F(\tau, \theta, x, \frac{dx}{dt})$ is a function periodic in θ with the period 2π, which may be represented in the form of the sum

$$F\left(\tau, \theta, x, \frac{dx}{dt}\right) = \sum_{n=-N}^{N} e^{in\theta} F_n\left(\tau, x, \frac{dx}{dt}\right), \qquad (18.2)$$

the coefficients of this finite sum $F_n(\tau, x, \frac{dx}{dt})$ being, in turn, certain polynomials $x, \frac{dx}{dt}$ whose coefficients depend on τ. We will assume, in addition, that $\frac{d\theta}{dt} = \nu(\tau)$, i.e., that the instantaneous frequency of the external periodic force likewise varies slowly with time.

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The construction of the approximate solutions of eq.(18.1) involves no additional fundamental difficulties and may be effected by means of the asymptotic method described above. We note that for the construction of the asymptotic series it is necessary that the coefficients of eq.(18.1), $m(\tau)$, $k(\tau)$, as well as $F_n(\tau, \frac{dx}{dt}, x)$ and $\nu(\tau)$ have a sufficient number of derivatives with respect to τ for all finite values of τ and, in addition, for any values of τ over the interval $0 < \tau < L$, $m(\tau) \neq 0$, $k(\tau) \neq 0$.

Under these assumptions, we may construct approximate solutions for eq.(18.1) in the most general form, suitable for investigations of the resonance zone and the approaches to it and for the case of any submultiple resonance.

Starting from the structure of the solution of eq.(12.1), it is natural, for eq.(18.1) under consideration, to seek the solution in the form of the series

$$x = a\cos\left(\frac{p}{q}\theta + \vartheta\right) + \varepsilon u_1\left(\tau, a, \theta, \frac{p}{q}\theta + \vartheta\right) + \varepsilon^2 u_2\left(\tau, a, \theta, \frac{p}{q}\theta + \vartheta\right) + \ldots, \tag{18.3}$$

in which $u_1(\tau, a, \theta, \frac{p}{q}\theta + \vartheta)$, $u_2(\tau, a, \theta, \frac{p}{q}\theta + \vartheta), \ldots$ are periodic functions of the angle θ and $\frac{p}{q}\theta + \vartheta$ with the period 2π; p and q, as above, are certain small relatively prime numbers, whose selection depends on the resonance we intend to investigate; the quantities a and ϑ are functions of time, which functions must be determined from the following system of differential equations:

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(\tau, a, \vartheta) + \varepsilon^2 A_2(\tau, a, \vartheta) + \ldots, \\ \frac{d\vartheta}{dt} &= \omega(\tau) - \frac{p}{q}\nu(\tau) + \varepsilon B_1(\tau, a, \vartheta) + \varepsilon^2 B_2(\tau, a, \vartheta) + \ldots, \end{aligned}\right\} \tag{18.4}$$

where $\omega(\tau) = \sqrt{\frac{k(\tau)}{m(\tau)}}$, is the "natural" frequency of the system while $\frac{d\theta}{dt} = \nu(\tau)$ denotes the instantaneous frequency of the external periodic disturbance.

Passing to a determination of the functions entering into the right sides of the expansions (18.3) and (18.4), we note that the expressions for them may be written on the basis of eqs.(13.35), (13.37), and (13.38), taking into account in the latter the slow variation of the mass $m(\tau)$, the coefficient of rigidity $k(\tau)$, the frequency of the external force $\nu(\tau)$ and the presence of "slow" time in the

right side of the equation.

After calculations analogous to those given in Section 13, but allowing for the insignificant variations introduced by the presence of τ, we find the approximate solution for eq.(18.1).

In first approximation, the solution of eq.(18.1) will have the form

$$x = a\cos\left(\frac{p}{q}\psi + \vartheta\right), \tag{18.5}$$

where a and θ must be determined from the system of equations of the first approximation:

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(\tau,\ a,\ \vartheta), \\ \frac{d\vartheta}{dt} &= \omega(\tau) - \frac{p}{q}\nu(\tau) + \varepsilon B_1(\tau,\ a,\ \vartheta), \end{aligned}\right\} \tag{18.6}$$

in which $A_1(\tau,\ a,\ \theta)$ and $B_1(\tau,\ a,\ \theta)$ are partial solutions, periodic in θ, of the system

$$\left.\begin{aligned} &\left[\omega(\tau) - \frac{p}{q}\nu(\tau)\right]\frac{\partial A_1}{\partial\vartheta} - 2a\omega(\tau)B_1 = \frac{1}{2\pi^2 m(\tau)}\sum_{\sigma} e^{i\sigma q\psi}\times \\ &\times\int_0^{2\pi}\int_0^{2\pi} F_0\left(\tau,\ a,\ \psi,\ \frac{p}{q}\psi + \vartheta\right)e^{-i\sigma q\psi}\cos\left(\frac{p}{q}\psi + \vartheta\right)d\psi\, d\left(\frac{p}{q}\psi + \vartheta\right), \\ &\left[\omega(\tau) - \frac{p}{q}\nu(\tau)\right]a\frac{\partial B_1}{\partial\vartheta} + 2\omega(\tau)A_1 = -\frac{1}{m(\tau)}\frac{d\,[m(\tau)\omega(\tau)]}{d\tau} - \\ &-\frac{1}{2\pi^2 m(\tau)}\sum_{\sigma} e^{i\sigma q\psi}\times \\ &\times\int_0^{2\pi}\int_0^{2\pi} F_0\left(\tau,\ a,\ \psi,\ \frac{p}{q}\psi + \vartheta\right)e^{-i\sigma q\psi}\sin\left(\frac{p}{q}\psi + \vartheta\right)d\psi\, d\left(\frac{p}{q}\psi + \vartheta\right), \end{aligned}\right\} \tag{18.7}$$

in which, as usual, the notation

$$F_0\left(\tau, a, \psi, \frac{p}{q}\psi + \vartheta\right) = F\left(\tau, \psi, a\cos\left(\frac{p}{q}\psi + \vartheta\right), -a\omega\sin\left(\frac{p}{q}\psi + \vartheta\right)\right).$$

has been introduced.

In second approximation we have

$$x = a\cos\left(\frac{p}{q}\psi + \vartheta\right) + \varepsilon u_1\left(\tau,\ a,\ \psi,\ \frac{p}{q}\psi + \vartheta\right), \tag{18.8}$$

where a and θ are solutions of the following systems of equations:

$$\left.\begin{aligned}\frac{da}{dt} &= \varepsilon A_1(\tau, a, \vartheta) + \varepsilon^2 A_2(\tau, a, \vartheta),\\ \frac{d\vartheta}{dt} &= \omega(\tau) - \frac{p}{q}\nu(\tau) + \varepsilon B_1(\tau, a, \vartheta) + \varepsilon^2 B_2(\tau, a, \vartheta).\end{aligned}\right\} \tag{18.9}$$

Here $u_1\left(\tau, a, \theta, \frac{p}{q}\theta + \vartheta\right)$ is determined by the formula

$$u_1\left(\tau, a, \theta, \frac{p}{q}\theta + \vartheta\right) = \frac{1}{4\pi^2 m(\tau)} \sum_{n,m} \frac{e^{i\left[n\theta + m\left(\frac{p}{q}\theta + \vartheta\right)\right]}}{\omega^2 - (m\omega + n\nu)^2} \times \tag{18.10}$$

$$\times \int_0^{2\pi}\int_0^{2\pi} F_0\left(\tau, a, \theta, \frac{p}{q}\theta + \vartheta\right) e^{-i\left[n\theta + m\left(\frac{p}{q}\theta + \vartheta\right)\right]} d\theta\, d\left(\frac{p}{q}\theta + \vartheta\right),$$

in which the summing is performed for the values n, m, satisfying the condition $nq + p(m \pm 1) \neq 0$. In this case, the right side of eq. (18.10) will not contain terms whose denominators could vanish for any values of τ over the interval $0 \leq \tau \leq L$. Then, $A_2(\tau, a, \theta)$ and $B_2(\tau, a, \theta)$ are determined from the system of equations

$$\left.\begin{aligned}&\left[\omega(\tau) - \frac{p}{q}\nu(\tau)\right]\frac{\partial A_2}{\partial\vartheta} - 2a\omega(\tau)B_2 =\\ &= -\left[\frac{\partial A_1}{\partial a}A_1 + \frac{\partial A_1}{\partial\vartheta}B_1 + \frac{\partial A_1}{\partial\tau} - aB_1^2 + \frac{dm(\tau)}{d\tau}\frac{A_1}{m(\tau)}\right] + \frac{1}{2\pi^2 m(\tau)}\sum_\sigma e^{i\sigma q\vartheta} \times\\ &\times \int_0^{2\pi}\int_0^{2\pi} F_1\left(\tau, a, \theta, \frac{p}{q}\theta + \vartheta\right) e^{-i\sigma q\vartheta}\cos\left(\frac{p}{q}\theta + \vartheta\right) d\theta\, d\left(\frac{p}{q}\theta + \vartheta\right),\\ &\left[\omega(\tau) - \frac{p}{q}\nu(\tau)\right] a\frac{\partial B_2}{\partial\vartheta} + 2\omega(\tau)A_2 =\\ &= -\left[\frac{\partial B_1}{\partial a}aA_1 + \frac{\partial B_1}{\partial\vartheta}aB_1 + \frac{\partial B_1}{\partial\tau}a + 2A_1B_1 + \frac{dm(\tau)}{d\tau}\frac{aB_1}{m(\tau)}\right] -\\ &- \frac{1}{2\pi^2 m(\tau)}\sum_\sigma e^{i\sigma q\vartheta} \times\\ &\times \int_0^{2\pi}\int_0^{2\pi} F_1\left(\tau, a, \theta, \frac{p}{q}\theta + \vartheta\right) e^{-i\sigma q\vartheta}\sin\left(\frac{p}{q}\theta + \vartheta\right) d\theta\, d\left(\frac{p}{q}\theta + \vartheta\right),\end{aligned}\right\} \tag{18.11}$$

where $F_1\left(\tau, a, \theta, \frac{p}{q}\theta + \vartheta\right)$ is a function periodic in θ and $\frac{p}{q}\theta + \vartheta$ with period 2π, the explicit expression for which becomes known as soon as we find $u_1\left(\tau, a, \theta, \frac{p}{q}\theta + \vartheta\right)$.

We note that for eqs.(1.1) or (12.1) as well as for eq.(18.1), the construction of approximate solutions makes use of the equations of harmonic balance, which, in the case under consideration take the following form:

$$\left.\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}\left\{\frac{d}{dt}\left[m(\tau)\frac{dx}{dt}\right]+k(\tau)x-\right.\\&\qquad\left.-\varepsilon F\left(\tau,\ \theta,\ x,\ \frac{dx}{dt}\right)\right\}\Bigg|_{x=a\cos\left(\frac{p}{q}\theta+\vartheta\right)+\dots}\times\\&\qquad\times\cos\left(\frac{p}{q}\theta+\vartheta\right)d\theta\,d\left(\frac{p}{q}\theta+\vartheta\right)=0,\\&\int_0^{2\pi}\int_0^{2\pi}\left\{\frac{d}{dt}\left[m(\tau)\frac{dx}{dt}\right]+k(\tau)x-\right.\\&\qquad\left.-\varepsilon F\left(\tau,\ \theta,\ x,\ \frac{dx}{dt}\right)\right\}\Bigg|_{x=a\cos\left(\frac{p}{q}\theta+\vartheta\right)+\dots}\times\\&\qquad\times\sin\left(\frac{p}{q}\theta+\vartheta\right)d\theta\,d\left(\frac{p}{q}\theta+\vartheta\right)=0.\end{aligned}\right\}\tag{18.12}$$

On substituting in the integrand of this expression the values of x, $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ found from eq.(18.3) with an accuracy to terms of the first order of smallness [of course, taking into account the fact that a and θ are functions of time satisfying eq.(18.6)] and, on performing the integration, we obtain eq.(18.7) for the determination of $A_1(\tau,\ a,\ \theta)$ and $B_1(\tau,\ a,\ \theta)$.

Likewise taking account, in the substitution of x, $\frac{dx}{dt}$, and $\frac{d^2x}{dt^2}$, the quantities proportional to ε^2, we obtain eq.(18.11), determining the quantities $A_2(\tau,\ a,\ \theta)$ and $B_2(\tau,\ a,\ \theta)$.

We note that, in calculating the integrals of eq.(18.12), τ is considered a certain constant parameter.

Let us now consider various special cases of eq.(18.1), and investigate concrete examples by using the obtained formulas.

As our first special case, consider the oscillations of the system described by a differential equation of the form

$$\frac{d}{dt}\left[m(\tau)\frac{dx}{dt}\right]+k(\tau)x=\varepsilon f\left(\tau,\ x,\ \frac{dx}{dt}\right).\tag{18.13}$$

According to our earlier discussions, the solution of eq.(18.13) in first approximation will be:

$$x = a\cos\psi, \tag{18.14}$$

where a and ψ must be determined from the equations

$$\left.\begin{aligned}
\frac{da}{dt} &= -\frac{\varepsilon a}{2m(\tau)\omega(\tau)}\frac{d[m(\tau)\omega(\tau)]}{d\tau} - \\
&\quad -\frac{\varepsilon}{2\pi m(\tau)\omega(\tau)}\int_0^{2\pi} f_0(\tau, a, \psi)\sin\psi\,d\psi, \\
\frac{d\psi}{dt} &= \omega(\tau) - \frac{\varepsilon}{2\pi m(\tau)\omega(\tau)a}\int_0^{2\pi} f_0(\tau, a, \psi)\cos\psi\,d\psi, \\
f_0(\tau, a, \psi) &= f(\tau, a\cos\psi, -a\omega\sin\psi).
\end{aligned}\right\} \tag{18.15}$$

We note that, by using the notation of Section 7, eq.(7.4), the equations of first approximation may be written in the form of

$$\left.\begin{aligned}
\frac{da}{dt} &= -\frac{\varepsilon a}{2m(\tau)\omega(\tau)}\frac{d[m(\tau)\omega(\tau)]}{d\tau} - \delta_e(\tau, a)a, \\
\frac{d\psi}{dt} &= \omega_e(\tau, a),
\end{aligned}\right\} \tag{18.16}$$

where $\delta_e(\tau, a)$ and $\omega_e(\tau, a)$ are, respectively, the equivalent damping decrement and the equivalent frequency, which differ from the expressions presented in Section 7 only in the presence of "slow" time, and are determined by the expressions

$$\left.\begin{aligned}
\delta_e(\tau, a) &= \frac{\varepsilon}{2\pi m(\tau)\omega(\tau)a}\int_0^{2\pi} f_0(\tau, a, \psi)\sin\psi\,d\psi, \\
\omega_e^2(\tau, a) &= \omega^2(\tau) - \frac{\varepsilon}{\pi m(\tau)\omega(\tau)a}\int_0^{2\pi} f_0(\tau, a, \psi)\cos\psi\,d\psi.
\end{aligned}\right\} \tag{18.17}$$

For constructing the second approximation, according to eqs.(18.8) and (18.9), we find after a number of calculations,

$$x = a\cos\psi + \varepsilon u_1(\tau, a, \psi), \tag{18.18}$$

where $u_1(\tau, a, \psi)$ is determined by the following series:

$$u_1(\tau, a, \psi) = \frac{1}{2\pi k(\tau)}\sum_{(n\neq\pm1)}\frac{e^{in\psi}}{1-n^2}\int_0^{2\pi} f_0(\tau, a, \psi)e^{-in\psi}\,d\psi, \tag{18.19}$$

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while the values a and ψ must satisfy the equations of second approximation

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(\tau,\ a) + \varepsilon^2 A_2(\tau,\ a), \\ \frac{d\psi}{dt} &= \omega(\tau) + \varepsilon B_1(\tau,\ a) + \varepsilon^2 B_2(\tau,\ a), \end{aligned}\right\} \quad (18.20)$$

in which $A_1(\tau,\ a)$ and $B_1(\tau,\ a)$ are of the same form as in the equations of first approximation (18.15), while the following expressions are found for $A_2(\tau,\ a)$ and $B_2(\tau,\ a)$:

$$\left.\begin{aligned} A_2(\tau,\ a) &= -\frac{1}{2\omega(\tau)}\left\{a\frac{\partial B_1}{\partial a}A_1 + a\frac{\partial B_1}{\partial \tau} + 2A_1B_1 + \right. \\ &\left. + \frac{a}{m(\tau)}\frac{dm(\tau)}{d\tau}B_1\right\} - \frac{1}{2\pi m(\tau)\omega(\tau)}\int_0^{2\pi} f_1(\tau,\ a,\ \psi)\sin\psi\, d\psi, \\ B_2(\tau,\ a) &= \frac{1}{2a\omega(\tau)}\left\{\frac{\partial A_1}{\partial a}A_1 + \frac{\partial A_1}{\partial \tau} - aB_1^2 + \right. \\ &\left. + \frac{1}{m(\tau)}\frac{dm(\tau)}{d\tau}A_1\right\} - \frac{1}{2\pi m(\tau)\omega(\tau)}\int_0^{2\pi} f_1(\tau,\ a,\ \psi)\cos\psi\, d\psi, \end{aligned}\right\} \quad (18.21)$$

using the following notation

$$\begin{aligned} f_1(\tau,\ a,\ \psi) &= f'_x(\tau,\ a\cos\psi,\ -a\omega\sin\psi)\, u_1 + \\ &+ f'_{x'}(\tau,\ a\cos\psi,\ -a\omega\sin\psi)\left[A_1\cos\psi - aB_1\sin\psi + \frac{\partial u_1}{\partial\psi}\omega(\tau)\right]. \end{aligned}$$

A comparison of the expressions obtained for the first and second approximations with the results of Section 1, indicates that the general scheme of constructing the solutions for the case of an oscillatory system described by eq. (18.13) will be the same as in the case of the system considered in Section 1. The equations of first approximation obtained by us differ from eq. (1.24) by the presence of "slow" time and of the additional summand $-\frac{\varepsilon a}{2m(\tau)\omega(\tau)}\frac{d[m(\tau)\omega(\tau)]}{d\tau}$.

Thus, in first approximation, the slow variability of the mass and the coefficient of elasticity, besides disturbing the harmonic character of the oscillations, also introduces additional "forces of friction", whose sign will depend on the manner in which the parameters of the oscillatory system under investigation vary.

As an example, let us consider the oscillations of a mathematical pendulum in the presence of a small damping, proportional to the first power of the velocity, and of a slow variation in length. Denoting the angle of deflection of the pendulum from the vertical position by θ; the acceleration of gravity by g; the mass of the pendulum by m; the slowly-varying length by $l = l(\tau)$; the coefficient of friction by $2n$, we obtain the differential equation

$$\frac{d}{dt}\left[ml^2(\tau)\frac{d\theta}{d\tau}\right] + 2n\frac{d}{dt}[l(\tau)\theta] + mgl(\tau)\sin\theta = 0. \qquad (18.22)$$

For small deflections we may consider, instead of eq.(18.22), the equation

$$\frac{d}{dt}\left[ml^2(\tau)\frac{d\theta}{dt}\right] + mgl(\tau)\theta = \varepsilon f\left(\tau, \theta, \frac{d\theta}{dt}\right), \qquad (18.23)$$

in which

$$\varepsilon f\left(\tau, \theta, \frac{d\theta}{dt}\right) = \frac{mgl(\tau)}{6}\theta^3 - 2nl(\tau)\frac{d\theta}{dt} - 2\varepsilon n\frac{dl(\tau)}{d\tau}\theta. \qquad (18.24)$$

In first approximation, eqs.(18.14) and (18.15) furnish

$$\theta = a\cos\psi, \qquad (18.25)$$

where a and ψ must be determined from the equations

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{3\varepsilon l'(\tau)}{4l(\tau)}a - \frac{na}{ml(\tau)}, \\ \frac{d\psi}{dt} &= \omega(\tau) - \frac{\omega(\tau)a^2}{16}, \quad \left(\omega(\tau) = \sqrt{\frac{g}{l(\tau)}}\right). \end{aligned}\right\} \qquad (18.26)$$

Integrating the first equation of system (18.26) with the initial values $t = 0$, $a = a_0$, we obtain the following expression for a:

$$a = a_0 e^{-\frac{n}{m}\int_0^t \frac{dt}{l(\tau)}}\left(\frac{l(0)}{l(\tau)}\right)^{\frac{3}{4}} \qquad (18.27)$$

On substituting the value of a from eq.(18.27) in the second equation of the system (18.26), we find

$$\psi = \int_0^t \omega(\tau)\left(1 - \frac{a_0^2 e^{-\frac{2n}{m}\int_0^t \frac{dt}{l(\tau)}}\left(\frac{l(0)}{l(\tau)}\right)^{\frac{3}{2}}}{16}\right)dt. \qquad (18.28)$$

Equations (18.27) and (18.28) enable us to construct graphs of the time-dependence

of amplitude and phase when the length of the pendulum varies slowly by an arbitrary law.

If, in these formulas, we put l = const, we obtain

$$a = a_0 e^{-\frac{\lambda}{2} t}, \qquad (18.29)$$

$$\psi = \omega\left(t + \frac{a_0^2 (e^{-\lambda t} - 1)}{16\lambda}\right) + \varphi,$$

where $\lambda = \frac{2n}{ml}$, and φ is the initial phase value.

The latter equations coincide with eq.(2.44) already found.

We now assume that the length of the pendulum varies by the linear law $l(\tau) = l_0 + l_1\tau$ where l_0 is the length at $t = 0$, εl_1 is the rate of change of the pendulum length (for a short time interval we can always assume, with a sufficient degree of accuracy, that the length varies by a linear law). In this case, we have the following expressions for the amplitude and phase:

$$a = a_0 \left(\frac{l_0}{l_0 + l_1\tau}\right)^{\frac{3}{4} + \frac{n}{ml_1\varepsilon}}, \qquad (18.30)$$

$$\psi = \int_0^t \sqrt{\frac{g}{l_0 + l_1\tau}}\left[1 - \frac{a_0^2}{16}\left(\frac{l_0}{l_0 + l_1\tau}\right)^{\frac{3}{2} + \frac{2n}{ml_1\varepsilon}}\right] dt. \qquad (18.31)$$

According to eq.(18.30), the amplitude of the oscillations under slow changes in the pendulum length will vary not by an exponential law, as in ordinary linear friction, but will be inversely proportional to some power of a function of time. It is obvious here that, at $n < 0$, $l_1 > 0$ and $\left|\frac{n}{ml_1\varepsilon}\right| < \frac{3}{4}$, and also at $n > 0$ and $l_1 > 0$, the oscillations will decay.

Thus a slow increase in pendulum length, as was to be expected, favors damping of the oscillations. If $l_1 < 0$, $n > 0$, and $\left|\frac{n}{l_1 m\varepsilon}\right| < \frac{3}{4}$, then the amplitude will increase, while for $\left|\frac{n}{l_1 m\varepsilon}\right| > \frac{3}{4}$, the amplitude will decrease.

For $l_1 < 0$ and $n_1 < 0$, the amplitude will increase.

In the absence of damping ($n = 0$), the amplitude of oscillations will increase with decreasing length and decrease with increasing length.

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An analogous analysis may be made for the oscillation frequency. For example, in the absence of damping, the instantaneous frequency decreases with an increase in in the pendulum length and increases with a decrease in length.

Let us calculate the second approximation for this example: According to equations (18.18), (18.19), and (18.20), after a number of calculations, we have

$$x = a\cos\psi - \frac{a^3}{192}\cos 3\psi, \tag{18.32}$$

where a and ψ must be determined from the equations

$$\left.\begin{aligned} \frac{da}{dt} &= -\left(\frac{3l'(\tau)}{4l(\tau)}\varepsilon + \frac{n}{ml(\tau)}\right)\left(a + \frac{a^3}{16}\right), \\ \frac{d\psi}{dt} &= \omega(\tau) - \frac{\omega(\tau)a^2}{16} + \frac{1}{2\omega(\tau)}\left\{\frac{n^2}{m^2l^2(\tau)} + \right. \\ &\left. + \frac{\varepsilon l'(\tau)\,n}{ml^2(\tau)} + \frac{5\varepsilon^2 l''(\tau)}{4l(\tau)} + \frac{5\omega^2(\tau)a^4}{2^9\cdot 3} - \frac{3\varepsilon^2 l'^2(\tau)}{16l^2(\tau)}\right\}, \end{aligned}\right\} \tag{18.33}$$

which may likewise be totally integrated. Thus, the first equation of the system (18.33) furnishes the following relation between a and t:

$$\frac{a}{\sqrt{16 + a^2}} = \frac{a_0}{\sqrt{16 + a_0^2}}\, e^{-\frac{n}{m}\int_0^t \frac{dt}{l(\tau)}}\left(\frac{l(0)}{l(\tau)}\right)^{\frac{3}{4}}, \tag{18.34}$$

after which we can also integrate the second equation of the system (18.33).

As a second special case, consider the differential equation of the oscillations of a nonlinear vibrator under the influence of a sinusoidal force, whose amplitude and instantaneous frequency are varying slowly. In this case, we have the following differential equation:

$$m\frac{d^2x}{dt^2} + kx = \varepsilon f\left(x, \frac{dx}{dt}\right) + \varepsilon E(\tau)\sin\theta, \tag{18.35}$$

where $\frac{d\theta}{dt} = \nu(\tau)$; $\tau = \varepsilon t$, while m and k are constants.

The oscillatory systems described by equations of this type play an important role in machine-building, electrical engineering, etc.

As indicated above, in such systems, in first approximation, we are able to detect only the fundamental resonance; therefore, making use of eq.(18.6), let us set up the equation of first approximation for the case of the fundamental resonance $p = 1$, $q = 1$.

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After a number of calculations for the solution of eq.(18.35) in first approximation, we obtain

$$x = a\cos(\theta + \vartheta), \tag{18.36}$$

where a and θ must be determined from the system of equations

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon}{2\pi m\omega}\int_0^{2\pi} f_0(a,\ \theta+\vartheta)\sin(\theta+\vartheta)\,d(\theta+\vartheta) - \\ &\qquad - \frac{\varepsilon E(\tau)}{m[\omega+\nu(\tau)]}\cos\vartheta, \\ \frac{d\vartheta}{dt} &= \omega - \nu(\tau) - \frac{\varepsilon}{2\pi m\omega a}\int_0^{2\pi} f_0(a,\theta+\vartheta)\cos(\theta+\vartheta)\,d(\theta+\vartheta) + \\ &\qquad + \frac{\varepsilon E(\tau)}{ma[\omega+\nu(\tau)]}\sin\vartheta, \end{aligned}\right\} \tag{18.37}$$

where

$$f_0(a,\ \theta+\vartheta) = f(a\cos(\theta+\vartheta),\ -a\omega\sin(\theta+\vartheta)).$$

Making use of the notation of eq.(14.5) (cf. Section 14), the system (18.37) may be represented in the following form:

$$\left.\begin{aligned} \frac{da}{dt} &= -\delta_e(a)\,a - \frac{\varepsilon E(\tau)}{m[\omega+\nu(\tau)]}\cos\vartheta, \\ \frac{d\vartheta}{dt} &= \omega_e(a) - \nu(\tau) + \frac{\varepsilon E(\tau)}{ma[\omega+\nu(\tau)]}\sin\vartheta, \end{aligned}\right\} \tag{18.38}$$

where $\delta_e(a)$ and $\omega_e(a)$ are, respectively, the equivalent damping decrement and the equivalent frequency for the nonlinear oscillatory system described by eq.(1.1).

From eq.(18.10) we find the expression for $u_1(\tau,\ a,\ \theta,\ \theta+\vartheta)$:

$$\begin{aligned} u_1(\tau,\ a,\ \theta,\ \theta+\vartheta) &= \frac{1}{\pi\omega^2}\sum_{n=2}^{N}\frac{1}{1-n^2}\Big[\cos n(\theta+\vartheta)\ \times \\ &\times \int_0^{2\pi} f_0(a,\ \theta+\vartheta)\cos n(\theta+\vartheta)\,d(\theta+\vartheta) + \\ &+ \sin n(\theta+\vartheta)\int_0^{2\pi} f_0(a,\ \theta+\vartheta)\sin(\theta+\vartheta)\,d(\theta+\vartheta)\Big], \end{aligned} \tag{18.39}$$

which does not depend on the slow time τ and coincides with the second summand in the right side of eq.(14.4).

We will not develop the expressions for $A_2(\tau,\ a,\ \theta)$ and $B_2(\tau,\ a,\ \theta)$, and merely

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remark that to determine them, after we have found the value of $u_1(\tau, a, \theta, \theta + \vartheta)$, it is simplest to make use of the equations of harmonic balance (18.12), taking into account the above remark on this subject.

Let us apply these formulas in investigating the oscillations on passage through a resonance in a concrete system. To make it easier to compare the results obtained from the stationary state studied, let us consider, as in Section 14, a nonlinear vibrator with a hard characteristic of the nonlinear restoring force ($F = cx + dx^3$), under the influence of an external sinusoidal force of constant amplitude and variable frequency. Let the oscillations of this vibrator be described by the following equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx + dx^3 = E\sin\theta, \tag{18.40}$$

where x denotes a coordinate determining the position of the system; t the time; m the mass; b the coefficient of resistance; $F = cx + dx^3$ the nonlinear restoring elastic force; E the amplitude of the disturbing force; $\theta(t)$ a certain function of time. To simplify the calculations, let us introduce, as above, the dimensionless x_1 and t_1 by the formulas

$$x_1 = \sqrt{\frac{d}{c}}\,x, \quad t_1 = \sqrt{\frac{c}{m}}\,t; \tag{18.41}$$

Then, eq.(18.40) will be written in the form

$$\frac{d^2x_1}{dt_1^2} + \delta\frac{dx_1}{dt_1} + x_1 + x_1^3 = E_1\sin\theta, \tag{18.42}$$

where

$$\delta = \frac{b}{\sqrt{mc}}, \quad E_1 = \frac{E}{c}\sqrt{\frac{d}{c}}.$$

Assume that the friction, the amplitude of the external force, and the term characterizing the nonlinearity are sufficiently small by comparison with the natural frequency of the system, i.e., that the system is a close-to-linear conservative system, and put:

$$-\left[\delta\frac{dx_1}{dt_1} + x_1^3\right] = \varepsilon f\left(x_1, \frac{dx_1}{dt_1}\right). \tag{18.43}$$

Then eq.(14.5) yields $\delta_e(a) = -\frac{\delta}{2}$; $k_e(a) = 1 + \frac{3a^2}{8}$, after which, in first approximation, we have

$$x_1 = a\cos(\theta + \vartheta),$$

where a and ϑ must be determined from the system of equations:

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\delta a}{2} - \frac{E_1}{1+\nu(\tau)}\cos\vartheta, \\ \frac{d\vartheta}{dt} &= 1 - \nu(\tau) + \frac{3a^2}{8} + \frac{E}{a[1+\nu(\tau)]}\sin\vartheta, \end{aligned}\right\} \quad (18.44)$$

where $\nu(\tau) = \frac{d\theta}{dt}$ is a certain function of time characterizing the law of variation of the instantaneous frequency of the external force with time.

In Section 14 we examined in detail the stationary state of systems described by an equation of the type of eq.(18.40), constructed resonance curves, investigated the stability of the various branches of the curves and considered the hysteresis phenomena arising in connection with the nonlinearity.

Here we will consider the behavior of the curves of the oscillation amplitude as a function of the frequency of the external force, during a slow variation in frequency with time; we will assume that, during this variation, the frequency of the external force passes through resonant values*. In order to construct the resonance curves on passage through resonance, the system of equations of first approximation (18.44) must be numerically integrated by some method of numerical integration. For the equations (18.44) under investigation, the method of numerical integration developed by A.N.Krylov is convenient. We note that there is no need of numerically integrating eq.(18.44) over the whole time interval during which the frequency of the external force varies. To obtain a complete picture of the process taking place on passage through resonance, it is sufficient to integrate the system (18.44) from an instant of time at which the frequency of the external force is sufficiently close to the natural frequency of the system, but has not yet reached the resonance zone directly. Practical constructions of resonance curves, on passage through a resonance show that, for values of the frequency of the external force at

*This question has been discussed by us in an earlier paper (Bibl.29)

which the stationary resonance curve is close to a horizontal line, the curves of passage through resonance differ little from the stationary resonance curves, even

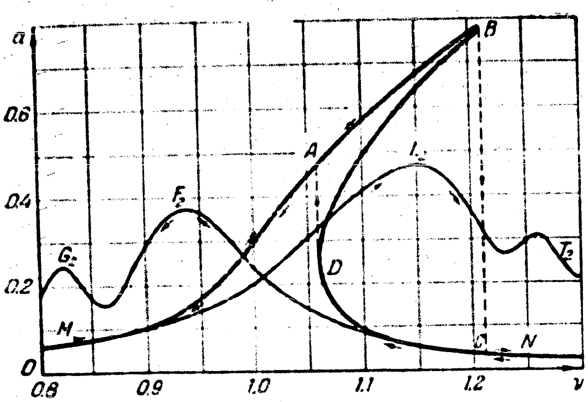

Fig. 109

when the rate of change of the frequency of the external force is fairly high. In addition, the initial values have almost no effect on the character of the resonance

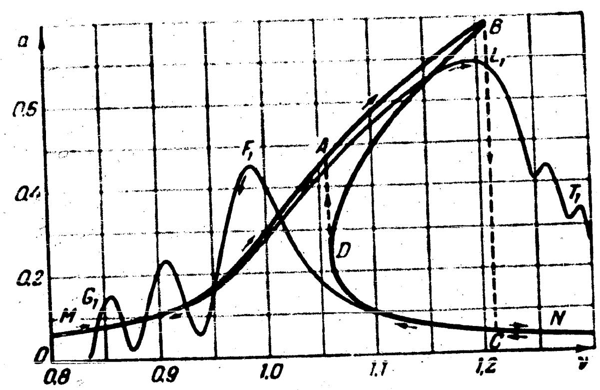

Fig. 110

curve as it passes through resonance (on the value and position of the maximum, etc.), provided they are not immediately within the resonance zone itself (i.e., in the

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zone of frequencies where the amplitude sharply rises). For this reason, for a numerical integration of the system (18.44) it is expedient to adopt, as the initial values, the values of a, θ and ν that satisfy the stationary state close to the resonant zone, but are not yet in the zone of rapidly increasing amplitudes.

We note that eq.(18.40) could be directly integrated by means of numerical methods; however, this would be a complicated task, requiring an extraordinary amount of

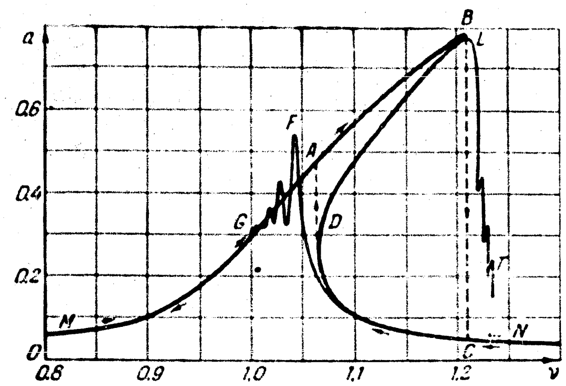

Fig.111

time and presenting much difficulty because of the possibility of cumulation of a large systematic error. However, a numerical integration of the equation of first (or second) approximation involves no difficulty, because of the fact that the variables in these equations are the amplitude and the phase.

To obtain a complete picture of the process, it is sufficient to calculate a small number of points located on a relatively "smooth" curve, which substantially simplifies the numerical integration, while the direct integration of eq.(18.40) would require us to find the sinusoid directly, instead of the envelope.

For simplicity we will consider the case when the instantaneous frequency of the external force depends linearly on the time

$$\nu(\tau) = \nu_0 + \gamma t; \qquad (18.45)$$

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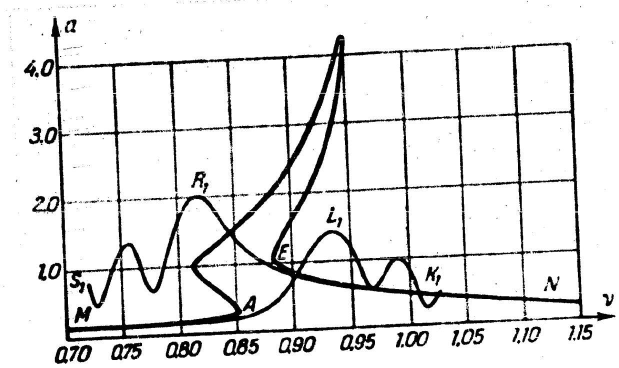

Fig. 112

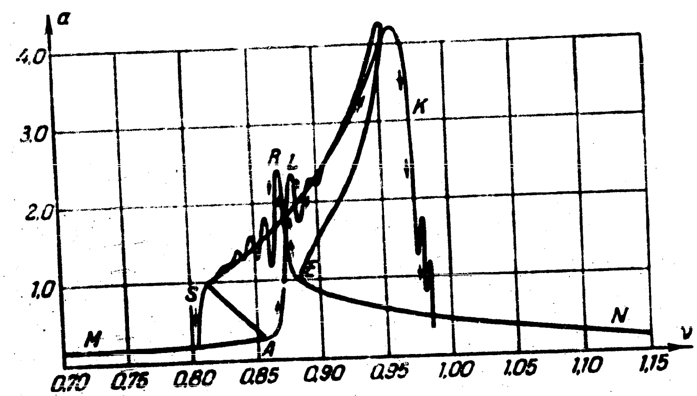

Fig. 113

For $\beta > 0$, the frequency increases with time and for $\beta < 0$, it decreases with time. The rate of passage through the resonance depends on the value of β. The larger the absolute value of β, the more rapidly the system will pass through resonance.

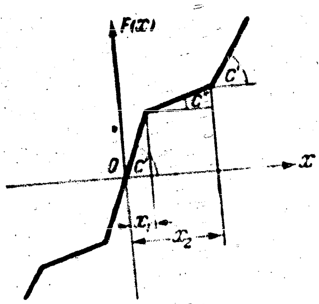

Fig.114

On numerical integration of the system of equations at various values of β, we obtain a number of curves of passage through resonance, which are presented in Figs.109, 110, 111. For comparison with the stationary state, the stationary resonance curves constructed by the formulas of Section 14 are presented in these same diagrams.

Figures 112 and 113 give both the stationary resonance curves and the curves of passage through resonance for a case when the characteristic of the nonlinear restoring force has the form given in Fig.114.

Analysis of the resonance curves so constructed on their passage through resonance allows a number of characteristic features of this complex phenomenon to be elucidated, as well as the influence of the nonlinearity of the system on it. We will not, however, go into this question here, since it has been considered in the special literature (Bibl.29).

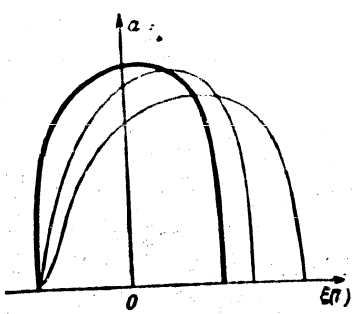

Fig.115

We will further discuss the investigation of certain examples of the linear oscillatory systems with variable coefficients, in which a more complex resonance is possible. As our first example, consider the behavior of the amplitude in resonance of the n^{th} kind depending on the state of variation of the detuning in an oscillatory circuit with vacuum-tube feedback. This example, in the case of $n = 2$ with constant detuning, has already been

considered in Section 14.

If the detuning $\xi = \frac{\nu^2 - 4\omega^2}{4\omega^2}$ varies with time, then it is obvious that instead of eq.(14.50), we will obtain the equation

$$\frac{d^2x}{dt^2} + x = \varepsilon(\tau) f\left(x, \frac{dx}{dt}, \xi(\tau)\right) + E \sin 2t, \tag{18.46}$$

which, by means of the substitution of variables

$$x = z - \frac{E}{3} \sin 2t \tag{18.47}$$

may be reduced to the form

$$\frac{d^2z}{dt^2} + z = \varepsilon(\tau) f\left(z - \frac{E}{3} \sin 2t,\ \frac{dz}{dt} - \frac{2E}{3} \cos 2t,\ \xi(\tau)\right). \tag{18.48}$$

Assume, as in Section 14, that

$$f\left(x, \frac{dx}{dt}, \xi(\tau)\right) = [k(\tau) + 2x + \gamma x^2] \frac{dx}{dt} + \frac{\xi(\tau)}{0{,}016} x, \tag{18.49}$$

and, to make it more definite, let us put

$$\left.\begin{aligned} k(\tau) &= k_0 - 2\overline{\vartheta} \frac{\xi(\tau)}{\beta}, \quad \varepsilon(\tau) = \frac{0.016}{1 + \xi(\tau)}, \quad \beta = 0{,}016, \\ \overline{\vartheta} &= 0{,}013, \quad \gamma = -2, \quad k_0 = -0{,}05. \end{aligned}\right\} \tag{18.50}$$

Then, eq.(18.48) takes the following form

$$\begin{aligned} \frac{d^2z}{dt^2} + z = \frac{0{,}016}{1 + \xi(\tau)} \Big\{ k(\tau) - 2\left(z - \frac{E}{3} \sin 2t\right) &+ \\ + \gamma \left(z - \frac{E}{3} \sin 2t\right)^2 \Big\} \left(\frac{dz}{dt} - \frac{2E}{3} \cos 2t\right) &+ \\ + \frac{\xi(\tau)}{1 + \xi(\tau)} \left(z - \frac{E}{3} \sin 2t\right)&. \end{aligned} \tag{18.51}$$

Making use of eqs.(18.5) and (18.6) and putting p = 1, q = 2, a number of calculations gives, in the first approximation,

$$z = a \cos(t + \theta), \tag{18.52}$$

where a and θ must be determined from the system of equations

$$\frac{da}{dt} = \varepsilon(\tau) \left\{ \frac{1}{2} a \left[k(\tau) + \frac{\gamma a^2}{4} \right] + \frac{\gamma E^2 a}{36} + \frac{aE}{6} \sin 2\theta \right\}, \tag{18.53}$$

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$$\frac{d\theta}{dt} = \varepsilon(\tau)\left\{-\frac{\xi(\tau)}{2p} + \frac{E}{6}\cos 2\theta\right\}.$$

For constructing graphs characterizing the variation in oscillation amplitude at the resonance of the second kind under various states of variation of the detuning $\xi(\tau)$, we must, as usual, numerically integrate the system (18.53).

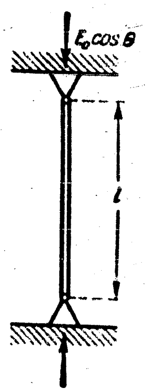

Fig. 116

To make the problem more definite, assume that the detuning varies as a function of the variation in the natural frequency of the original oscillatory system ω, and assume that $\xi(\tau)$ varies by

$$\xi(\tau) = \xi_0 + \alpha t. \tag{18.54}$$

Substituting the value of $\xi(\tau)$ of eq. (18.54) in equation (18.53) and performing numerical integration, we obtain curves characterizing the relation between a and ξ for various values of α (Fig. 115).

For comparison, the same diagram contains the resonance curve (heavy line) in the stationary state, constructed from eq. (14.72).

An analysis of the graph so obtained permits a number of conclusions: As usual, when the rate of passage through resonance increases, the maxima of the resonance curves are lowered and shifted. There is a striking difference in the behavior of the oscillation on passage through a resonance of the second kind by comparison with the above considered example of passage through ordinary resonance. While, in passage through ordinary resonance (cf. Fig. 110), the first maximum of the resonance curve is followed by a few more maxima of smaller values so that the fluctuations are of the nature of damped beats, the amplitude in our case, after reaching its maximum value, continues to decrease steadily and tends toward zero.

As our second example, consider the passage through parametric resonance. Let the rod of a length l with hinged ends (Fig. 116) be subjected to the "periodic" longitudinal force

$$E_0 \cos\theta \tag{18.55}$$

of instantaneous frequency $\frac{d\theta}{dt} = \nu(\tau)$, which varies slowly with time and passes through a doubled-critical value (to make the formulation more definite, assume it to be through the doubled first critical value).

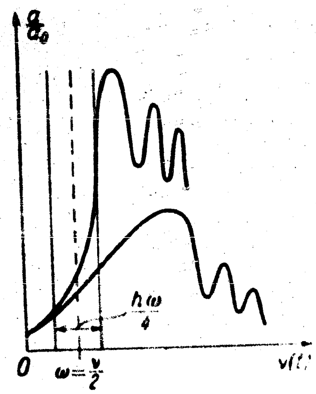

Fig. 117

The differential equation of the transverse vibrations of the rod may be written in the form

$$EJ\frac{\partial^4 y}{\partial z^4} + \frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} + E_0 \cos\theta \frac{\partial^2 y}{\partial z^2} = 0, \quad (18.56)$$

where, as in Section 16, A is the cross sectional area; EJ the rigidity; γ the density; and g the acceleration of gravity.

The boundary conditions will be

$$\left.\begin{array}{l} y\Big|_{z=0} = 0, \quad \frac{\partial^2 y}{\partial z^2}\Big|_{z=0} = 0, \\ y\Big|_{z=l} = 0, \quad \frac{\partial^2 y}{\partial z^2}\Big|_{z=l} = 0, \end{array}\right\} \quad (18.57)$$

Therefore, by means of the substitution

$$y = x \sin \pi \frac{z}{l}$$

eq.(18.56) can be reduced to the following:

$$\frac{d^2 x}{dt^2} + \omega^2 (1 - h\cos\theta)\, x = 0, \quad (18.58)$$

where

$$h = \frac{E_0 l^2}{EJ\pi^2}, \quad \omega^2 = \frac{EJ g \pi^4}{\gamma A l^4}.$$

Assuming that $\frac{d\theta}{dt} = \nu(t)$, varying with time, passes through the doubled values of the frequency ω, let us construct the first approximation corresponding to the resonance p = 1, q = 2.

Using eq.(18.5) and (18.6), we have

$$x = a\cos\left(\frac{1}{2}\theta + \vartheta\right), \quad (18.59)$$

where a and ϑ must be determined from the system

$$\frac{da}{dt} = -\frac{a h \omega^2}{2\nu(\tau)} \sin 2\vartheta, \quad (18.60)$$

$$\frac{d\theta}{dt} = \omega - \frac{\nu(\tau)}{2} - \frac{h\omega^2}{2\nu(\tau)}\cos 2\theta. \Bigg\}$$

On assigning numerical values to h, ω, $\nu(\tau) = \nu_o + \alpha t$, and integrating the system (18.60), we obtain the curves of passage through resonance shown in Fig.117.

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CHAPTER IV

THE METHOD OF THE MEAN

Section 19. Equations of First and Higher Approximations in the Method of the Mean

At the beginning of this book we had briefly discussed the reduction of a nonlinear differential equation (containing a small parameter) to the standard form and had described the construction of an approximate solution by the principle of averaging or the method of the mean.

This question will be discussed in more detail in this Chapter.

It is well known that the form of nonlinear differential equations containing a small parameter, and also the character of the small parameter itself, may vary widely.

In many cases, however, by means of simple substitutions of variables, the differential equations of oscillations may be reduced to one general form, in which the right sides are proportional to the small parameter. We decided to call this form of differential equations the standard form.

The reduction of differential equations to the standard form by application of the principle of averaging is an effective method, especially in studying nonlinear oscillatory systems with many degrees of freedom. Thus, for instance, in the case where a nonlinear oscillatory system with n degrees of freedom is characterized by the following expression for the kinetic and potential energies

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$$T = \frac{1}{2}\sum_{k,\,j=1}^{n} a_{kj}\dot{q}_k\dot{q}_j, \quad V = \frac{1}{2}\sum_{k,\,j=1}^{n} b_{kj}q_k q_j, \tag{19.1}$$

where $q_1, q_2, \ldots, q_n$ are generalized coordinates, a_{kj}, b_{kj} are constants, and the quadratic forms T and V are definitely positive, it is commonly known that the linear transformation

$$q_j = \sum_{k=1} \varphi_{jk} x_k \tag{19.2}$$

can be used for introducing the normal coordinates $x_1, x_2, \ldots,$ for which

$$T = \frac{1}{2}\sum_{k=1}^{n} \dot{x}_k^2, \quad V = \frac{1}{2}\sum_{k=1} \omega_k^2 x_k^2. \tag{19.3}$$

Then, the Lagrange equation for unperturbed motion takes the following form:

$$\frac{d^2 x_k}{dt^2} + \omega_k^2 x_k = 0 \quad (k = 1,\ 2,\ \ldots,\ n). \tag{19.4}$$

Assume now that our system is exposed to a small disturbance of the form

$$\varepsilon Q_k = \varepsilon \{Q_k^{(0)}(q_k, \dot{q}_k) + \sum_{\alpha} [Q_{k1}^{(\alpha)}(q_k, \dot{q}_k)\cos\Omega_\alpha t + Q_{k2}^{(\alpha)}(q_k, \dot{q}_k)\sin\Omega_\alpha t]\}, \tag{19.5}$$

where Ω_α denotes the frequency of the disturbing forces and ε is a small parameter.

Then, changing to normal coordinates in eq.(19.5) we obtain the following system of nonlinear equations:

$$\frac{d^2 x_k}{dt^2} + \omega_k^2 x_k = \varepsilon X_k(t,\ x_k,\ \dot{x}_k) \quad (k = 1,\ 2,\ \ldots,\ n), \tag{19.6}$$

where εX_k is determined from the condition of equivalence of work by the formula

$$X_k = \sum_{j=1}^{n} Q_j \varphi_{kj} \quad (k = 1, 2, \ldots, n). \tag{19.7}$$

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Equation (19.6), by substitution of the variables

$$x_k = z_k e^{i\omega_k t} + z_{-k} e^{-i\omega_k t}, \tag{19.8}$$

$$\dot{x}_k = i\omega_k z_k e^{i\omega_k t} - i\omega_k z_{-k} e^{-i\omega_k t}, \tag{19.9}$$

where z_k and z_{-k} are complex conjugate unknown functions of time, may be reduced to the standard form.

In fact, differentiating eq.(19.8) and comparing it with eq.(19.9), we get

$$\dot{z}_k e^{i\omega_k t} + z_{-k} e^{-i\omega_k t} = 0. \tag{19.10}$$

Differentiating eq.(19.9) and substituting in eq.(19.6), we obtain

$$i\omega_k \dot{z}_k e^{i\omega_k t} - i\omega_k z_{-k} e^{-i\omega_k t} = \varepsilon X_k. \tag{19.11}$$

Using, for a simplification of the notation,

$$-\omega_{-k} = \omega_k, \quad X_{-k} = X_k, \tag{19.12}$$

eq.(19.6) may be represented in the form

$$\frac{dz_g}{dt} = \varepsilon Z_g(t, z_k) \begin{pmatrix} g = \pm 1, \pm 2, \ldots, \pm n \\ k = \pm 1, \pm 2, \ldots, \pm n \end{pmatrix}. \tag{19.13}$$

Equations describing the oscillation of systems under the influence of forces of high frequency and of other systems, may also be reduced to equations of the type of eq.(19.13).

Let us, therefore, describe the formal method of constructing approximate solutions for equations in the standard form

$$\frac{dx_k}{dt} = \varepsilon X_k(t, x_1, x_2, \ldots, x_n) \quad (k = 1, 2, \ldots, n), \tag{19.14}$$

where ε is a small parameter and X_k may be represented by the sums

$$X_k(t, x_1, x_2, \ldots, x_n) = \sum_\nu e^{i\nu t} X_{k\nu}(x_1, x_2, \ldots, x_n) \tag{19.15}$$

$$(k = 1, 2, \ldots, n),$$

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CHAPTER IV

THE METHOD OF THE MEAN

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This question will be discussed in more detail in this Chapter.

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In many cases, however, by means of simple substitutions of variables, the differential equations of oscillations may be reduced to one general form, in which the right sides are proportional to the small parameter. We decided to call this form of differential equations the standard form.

The reduction of differential equations to the standard form by application of the principle of averaging is an effective method, especially in studying nonlinear oscillatory systems with many degrees of freedom. Thus, for instance, in the case where a nonlinear oscillatory system with n degrees of freedom is characterized by the following expression for the kinetic and potential energies

$$T = \frac{1}{2}\sum_{k,\,j=1}^{n} a_{kj}\dot{q}_k\dot{q}_j, \quad V = \frac{1}{2}\sum_{k,\,j=1}^{n} b_{kj}q_kq_j, \tag{19.1}$$

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$$\frac{d^2x_k}{dt^2} + \omega_k^2 x_k = 0 \quad (k = 1,\ 2,\ \ldots,\ n). \tag{19.4}$$

Assume now that our system is exposed to a small disturbance of the form

$$\varepsilon Q_k = \varepsilon\{Q_k^{(0)}(q_k, \dot{q}_k) + \sum_{\alpha}[Q_{k1}^{(\alpha)}(q_k, \dot{q}_k)\cos\Omega_\alpha t + Q_{k2}^{(\alpha)}(q_k, \dot{q}_k)\sin\Omega_\alpha t]\}, \tag{19.5}$$

where Ω_α denotes the frequency of the disturbing forces and ε is a small parameter.

Then, changing to normal coordinates in eq.(19.5) we obtain the following system of nonlinear equations:

$$\frac{d^2x_k}{dt^2} + \omega_k^2 x_k = \varepsilon X_k(t,\ x_k,\ \dot{x}_k) \quad (k = 1,\ 2,\ \ldots,\ n), \tag{19.6}$$

where εX_k is determined from the condition of equivalence of work by the formula

$$X_k = \sum_{j=1}^{n} Q_j\varphi_{kj} \quad (k = 1, 2, \ldots, n). \tag{19.7}$$

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In fact, differentiating eq.(19.8) and comparing it with eq.(19.9), we get

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Let us, therefore, describe the formal method of constructing approximate solutions for equations in the standard form

$$\frac{dx_k}{dt} = \varepsilon X_k(t, x_1, x_2, \ldots, x_n) \quad (k = 1, 2, \ldots, n), \tag{19.14}$$

where ε is a small parameter and X_k may be represented by the sums

$$X_k(t, x_1, x_2, \ldots, x_n) = \sum_\nu e^{i\nu t} X_{k\nu}(x_1, x_2, \ldots, x_n) \tag{19.15}$$

$$(k = 1, 2, \ldots, n),$$

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in which ν denotes constant frequencies.

It should be mentioned that eq.(19.14) is considered exclusively in the real region, and the complex form of representing the sinusoidal oscillations used in eq.(19.15) is introduced merely for simplicity of notation. In considering the high approximations, it is often expedient to consider terms of higher order with respect to ε in the differential equations. In this case we obtain, for example,

$$\frac{dx_k}{dt} = \varepsilon X_k(t, x_1, \ldots, x_n) + \varepsilon^2 Y_k(t, x_1, \ldots, x_n) + \cdots \quad (k = 1, 2, \ldots, n), \tag{19.16}$$

where Y_k is a function of the same form as X_k. This type of equation will also be denoted as the standard form. In applying the theory of perturbations, no substantial changes are introduced here.

Before proceeding to a description of this theory, we will introduce a number of abbreviated notations. Thus, the set of n quantities $x_1, x_2, \ldots, x_n$ will be denoted by the single letter x. Then eq.(19.14) will be written in the form

$$\frac{dx}{dt} = \varepsilon X(t, x), \tag{19.17}$$

where

$$X(t, x) = \sum_{\nu} e^{i\nu t} X_{\nu}(x). \tag{19.18}$$

The formulas for differentiation of complex functions

$$\frac{dF_k(t, x_1, \ldots, x_n)}{dt} = \frac{\partial F_k}{\partial t} + \sum_{q=1}^{n} \frac{\partial F_k}{\partial x_q}\frac{dx_q}{dt} \tag{19.19}$$

in our notation, will be

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{dx}{dt} = \frac{\partial F}{\partial t} + \left(\frac{dx}{dt}\frac{\partial}{\partial x}\right)F, \tag{19.20}$$

where, in this way, $\frac{\partial F}{\partial x}$ is treated as the matrix

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$$\left\| \frac{\partial F_k}{\partial x_q} \right\|,$$

applied to the vector $\frac{dx}{dt}$, while $\left(\frac{dx}{dt} \frac{\partial}{\partial x} \right)$ is treated as the scalar operator product

$$\sum_{q=1}^{n} \frac{dx_q}{dt} \frac{\partial}{\partial x_q}. \tag{19.21}$$

It is obvious that the use of this matrix-vector system of notation requires no particular explanation and has considerable advantages in shortening the formulas.

Let, further, F (t, x) be a sum of the form

$$F(t, x) = \sum_{\nu} e^{i\nu t} F_\nu(x). \tag{19.22}$$

Then, introducing the notation

$$\left. \begin{aligned} \underset{t}{M} \{F(t, x)\} &= F_0(x), \\ \widetilde{F}(t, x) &= \sum_{\nu \neq 0} \frac{e^{i\nu t}}{i\nu} F_\nu(x), \\ \widetilde{\widetilde{F}}(t, x) &= \sum_{\nu \neq 0} \frac{e^{i\nu t}}{(i\nu)^2} F_\nu(x), \end{aligned} \right\} \tag{19.23}$$

etc., we obtain identically

$$\frac{\partial \widetilde{\widetilde{F}}}{\partial t} = \widetilde{F}, \quad \frac{\partial \widetilde{F}}{\partial t} = F - \underset{t}{M} \{F\}. \tag{19.24}$$

We will denote the operator ~ as the integrating operator, the operator $\underset{t}{M}$ as the operator of averaging for constant x or the operator of averaging over explicity contained time.

Consider now the system of differential equations (19.17), where ε is a small parameter and where the expressions X, as functions of the time t, are represented

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by the sums (19.18).

We note that the form of the approximate solution may be found, or rather guessed, by entirely intuitive considerations, namely: Since the first derivatives $\frac{dx}{dt}$ are proportional to the small parameter, it is natural to consider all x as slowly varying quantities. Let us represent x as the superposition of a smoothly varying term ξ and a sum of small vibrational terms; in view of the smallness of these latter, we assume in first approximation that $x = \xi$. Then,

$$\frac{dx}{dt} = \varepsilon X(t,\ x) = \varepsilon X(t,\ \xi) = \varepsilon \sum_{\nu} X_{\nu}(\xi) e^{i\nu t}, \tag{19.25}$$

i.e.,

$$\frac{dx}{dt} = \varepsilon X_0(\xi) + \text{ small sinusoidal oscillatory terms} \tag{19.26}$$

Considering that these sinusoidal oscillatory terms are due only to the small vibrations of x about ξ and exert no influence on the systematic variation of x, we obtain the equation of first approximation in the form

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi) = \varepsilon \underset{t}{M} \{X(t,\ \xi)\}. \tag{19.27}$$

To obtain the second approximation it is also necessary to take the vibrational terms into consideration in the expression for x; assuming that the term $\varepsilon e^{i\nu t} X_{\nu}(\xi)$ in eq.(19.26) causes, in x, oscillations of the form

$$\frac{\varepsilon e^{i\nu t}}{i\nu} X_{\nu}(\xi),$$

we reach the following approximate expression:

$$x = \xi + \varepsilon \sum_{\nu \neq 0} \frac{e^{i\nu t}}{i\nu} X_{\nu}(\xi) = \xi + \varepsilon \widetilde{X}(t,\ \xi). \tag{19.28}$$

Substituting eq.(19.28) in eq.(19.17), we have

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$$\frac{dx}{dt} = \varepsilon X(t, \xi + \varepsilon\widetilde{X}), \qquad (19.29)$$

i.e.,

$$\frac{dx}{dt} = \varepsilon \underset{t}{M} \{X(t, \xi + \varepsilon\widetilde{X})\} + \text{small sinusoidal oscillatory terms}$$

whence, neglecting the influence of the sinusoidal oscillatory terms on the systematic variation of ξ, we obtain the equations of second approximation:

$$\frac{d\xi}{dt} = \varepsilon \underset{t}{M} \{X(t, \xi + \varepsilon\widetilde{X})\} = \varepsilon \underset{t}{M} \left\{X(t, \xi) + \varepsilon\left(\widetilde{X}\frac{\partial}{\partial\xi}\right) X(t, \xi)\right\} \qquad (19.30)$$

and so on.

This reasoning obviously cannot claim to be at all convincing; the objection can be raised, that, in setting up the approximate equations (19.27), terms of the same order of smallness as the retained term εX_0 were rejected in eqs.(19.17).

It is not hard, however, to put them into a more justified form.

For this purpose let us perform a substitution of variables in eq.(19.17)

$$x = \xi + \varepsilon\widetilde{X}(t, \xi), \qquad (19.31)$$

where the ξ terms are regarded as new unknowns.

Differentiating eq.(19.31), we have

$$\frac{dx}{dt} = \frac{d\xi}{dt} + \varepsilon\frac{\partial\widetilde{X}(t, \xi)}{\partial\xi}\frac{d\xi}{dt} + \varepsilon\frac{\partial\widetilde{X}(t, \xi)}{\partial t}. \qquad (19.32)$$

However, in view of the properties (19.24) of the integrating operator,

$$\frac{\partial\widetilde{X}(t, \xi)}{\partial t} = X(t, \xi) - X_0(\xi).$$

Substituting eqs.(19.31) and (19.32) in eq.(19.17), we obtain

$$\frac{d\xi}{dt} + \varepsilon\frac{\partial\widetilde{X}(t, \xi)}{\partial\xi}\frac{d\xi}{dt} + \varepsilon X(t, \xi) - \varepsilon X_0(\xi) = \varepsilon X\{t, \xi + \varepsilon\widetilde{X}(t, \xi)\}$$

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or

$$\left\{1+\varepsilon\frac{\partial\tilde{X}}{\partial\xi}\right\}\frac{d\xi}{dt}=\varepsilon X_0(\xi)+\varepsilon\{X(t,\xi+\varepsilon\tilde{X})-\tilde{X}(t,\xi)\}, \quad (19.33)$$

where 1 is regarded as the unit matrix.

Multiplying eq. (19.33) from the left by

$$\left\{1+\varepsilon\frac{\partial\tilde{X}}{\partial\xi}\right\}^{-1}, \quad (19.34)$$

we note that the new unknown ξ terms satisfy equations of the form

$$\frac{d\xi}{dt}=\varepsilon\left\{1+\varepsilon\frac{\partial\tilde{X}}{\partial\xi}\right\}^{-1}X_0(\xi)+ \\ +\varepsilon\left\{1+\varepsilon\frac{\partial\tilde{X}}{\partial\xi}\right\}^{-1}\{X(t,\xi+\varepsilon\tilde{X})-X(t,\xi)\}. \quad (19.35)$$

On the other hand, expanding eq. (19.34) into a power series of ε, we have

$$\left\{1+\varepsilon\frac{\partial\tilde{X}}{\partial\xi}\right\}^{-1}=1-\varepsilon\frac{\partial\tilde{X}(t,\xi)}{\partial\xi}+\varepsilon^2\ldots,$$

where, in general, the symbol ε^m denotes quantities of the order of smallness of ε^m. Equation (19.35) thus yields:

$$\frac{d\xi}{dt}=\varepsilon X_0(\xi)+\varepsilon^2\ldots \quad (19.36)$$

or, in more detail,

$$\frac{d\xi}{dt}=\varepsilon X_0(\xi)-\varepsilon^2\frac{\partial\tilde{X}(t,\xi)}{\partial\xi}X_0(\xi)+\varepsilon\{X(t,\xi+\varepsilon\tilde{X})-X(t,\xi)\}+\varepsilon^3\ldots= \\ =\varepsilon X_0(\xi)-\varepsilon^2\frac{\partial\tilde{X}(t,\xi)}{\partial\xi}X_0(\xi)+\varepsilon^2\left(\tilde{X}\frac{\partial}{\partial\xi}\right)X(t,\xi)+\varepsilon^3\ldots \quad (19.37)$$

Thus if ξ satisfies eq. (19.36), whose right side differs from the right side of the equation

$$\frac{d\xi}{dt}=\varepsilon X_0(\xi) \quad (19.38)$$

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by terms of the second order of smallness, then the expression

$$x = \xi + \varepsilon \tilde{X}(t, \xi) \qquad (19.39)$$

represents the exact solution of eq. (19.17) under consideration.

For this reason, we may take as our first approximation

$$x = \xi, \qquad (19.40)$$

taking as ξ the solution of the equations of first approximation (19.38).

Equation (19.39), in which ξ satisfies these same equations, will be called the refined first approximation.

Substituting the refined first approximation in the exact equation (19.17), it is obvious that this approximation satisfies them with an accuracy to terms of the second order of smallness.

It is clear that, for an effective construction of the approximate solution, it is primarily necessary to solve the equation of first approximation; the fact that these equations (like the exact equations) are differential equations, imposes a certain limitation on the possibility of using the described method. It must, however, be emphasized that, for a large number of cases of practical interest, the equations of first approximation are found to be far simpler and far more amenable to investigation. In many cases in which it is impossible to obtain a general solution, we may find at least important partial solutions, for example, the corresponding steady oscillatory processes.

For example, at $n = 1$ the equations of first approximation are integrated in quadratures; at $n = 2$ the famous Poincaré theory may be used for their investigation.

For any value of n, if $X_0(\xi)$ vanished at a certain point $\xi = \xi_0$, we may consider the "quasi-static" solution

$$x = \xi_0$$

of the equations of first approximation. To investigate the stability of this solution, we can proceed in the usual way, by setting up the equations for small deviations (equations of variation);

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$$\frac{d\delta\xi}{dt} = \varepsilon \frac{\partial X_0(\xi_0)}{\partial \xi} \delta\xi. \qquad (19.41)$$

If all the real parts of the roots of the characteristic equation

$$\mathrm{Det}\left| 1 \cdot p - \varepsilon \frac{\partial X_0(\xi_0)}{\partial \xi} \right| = 0 \qquad (19.42)$$

are negative, then the quasi-static solution under consideration will be stable. Every solution of the equations of first approximation, starting from initial values sufficiently close to ξ, will exponentially approach the quasi-static solution as $t \to \infty$. If, even for one of the roots of the characteristic equation, the real part is positive, we have a case of instability. We may also represent the critical case, when all the real parts are equal to zero. This case can sometimes be reduced to the two preceding cases by considering higher approximations.

As shown by the refined first approximation for the quasi-static solution under consideration, x is represented in the form of the sum of a constant term and small sinusoidal oscillations with the "external" frequencies ν. The higher approximations would likewise reveal the presence of terms with compound frequencies consisting of the frequencies ν.

These conclusions, formulated on consideration of the approximate solutions, can be confirmed for the exact solutions of eq.(19.17), on the basis of rigorous mathematical theory. It has been shown (Bibl.6) that, in the case where the real parts of the roots of the characteristic equation (19.24) do not vanish, we may establish, for very general conditions, that the exact equations (19.17) have a quasi-periodic solution $x = x(t)$ (with base frequencies ν), lying in the vicinity of the point $x = \xi$. This area may be taken as small as desired for sufficiently small values of ε. This quasi-periodic solution will be stable or unstable, according to the signs of real parts of the roots of the algebraic equation (19.42).

Returning to eq.(19.38), we note that, by definition of the averaging operator,

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we have

$$X_0(\xi) = \varepsilon \underset{t}{M}\{X(t, \xi)\}$$

Consequently, the equations of first approximation may be represented in the form

$$\frac{d\xi}{dt} = \varepsilon \underset{t}{M}\{X(t, \xi)\}. \qquad (19.43)$$

Thus the equations of first approximation (19.43) are obtained from the exact equations (19.17) by averaging the latter over the explicitly contained time t. In forming the mean, the ξ terms are treated as constants.

This formal process, consisting in the replacement of exact equations by averaged ones, is sometimes called the principle of averaging or the method of the mean.

As demonstrated later, for justifying the method of the mean it is not required that X (t, ξ) must be representable by the sum (19.18); what is of substantial importance here is only the existence of the mean value:

$$X_0(\xi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t, \xi)\,dt. \qquad (19.44)$$

It must be noted that, in one form or another, the method of the mean has long been used for obtaining approximate solutions. Thus, as far back as the method of "secular perturbations" developed by the founders of celestial mechanics, practically the same method of the mean had been in use. However, it was only recently that mathematicians have begun to concern themselves with the problem of justifying this principle.

Let us describe below the construction of the second approximation.

We note that, in the construction of the first approximation by substitution of the variables (19.31), eq. (19.17) was transformed into

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi) + \varepsilon^2 \ldots$$

To obtain the second approximation we find the analogous substitution of

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variables, transforming the variable x to ξ, satisfying an equation of the form

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi) + \varepsilon^2 P(\xi) + \varepsilon^3 \ldots \qquad (19.45)$$

To apply this substitution of variables by what (in our opinion) is the most natural method, let us find the expression

$$x = \Phi(t, \xi, \varepsilon), \qquad (19.46)$$

which, for values of ξ satisfying an equation of the type

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi) + \varepsilon^2 P(\xi), \qquad (19.47)$$

would satisfy eq. (19.17) with an accuracy to terms of the order of smallness of ε^3.

Since, at ξ determined from the equation of first approximation

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi),$$

the expression

$$x = \xi + \varepsilon \sum_{\nu \neq 0} \frac{e^{i\nu t}}{i\nu} X_\nu(\xi) = \xi + \varepsilon \tilde{X}(t, \xi)$$

will satisfy eq. (19.17) with an accuracy to terms of the order of smallness of ε^2, we will look for the solution of eq. (19.46) in the form

$$x = \xi + \varepsilon \tilde{X}(t, \xi) + \varepsilon^2 F(t, \xi), \qquad (19.48)$$

where F is represented by sums of the form

$$F(t, \xi) = \sum_{\mu} e^{i\mu t} F_\mu(\xi). \qquad (19.49)$$

However, for eq. (19.48),

$$\begin{aligned} \varepsilon X(t, x) &= \varepsilon X(t, \xi + \varepsilon \tilde{X}) + \varepsilon^2 \ldots = \\ &= \varepsilon X(t, \xi) + \varepsilon^2 \left(\tilde{X} \frac{\partial}{\partial \xi}\right) X(t, \xi) + \varepsilon^3 \ldots \end{aligned} \qquad (19.50)$$

On the other hand, for values of ξ determined from eq. (19.47), we find by

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differentiating eq. (19.48),

$$\frac{dx}{dt} = \frac{d\xi}{dt} + \varepsilon\frac{\partial \tilde{X}(t,\xi)}{\partial\xi}\frac{d\xi}{dt} + \varepsilon^2\frac{\partial F(t,\xi)}{\partial\xi}\frac{d\xi}{dt} + \varepsilon\frac{\partial \tilde{X}(t,\xi)}{\partial t} + \varepsilon^2\frac{\partial F(t,\xi)}{\partial t} = \varepsilon X_0(\xi) + \varepsilon^2 P(\xi) + \varepsilon^2\frac{\partial \tilde{X}(t,\xi)}{\partial\xi}X_0(\xi) + \varepsilon\frac{\partial \tilde{X}(t,\xi)}{\partial t} + \varepsilon^2\frac{\partial F(t,\xi)}{\partial t} + \varepsilon^3 \ldots,$$

whence

$$\frac{dx}{dt} = \varepsilon X(t,\xi) + \varepsilon^2 P(\xi) + \varepsilon^2\frac{\partial \tilde{X}(t,\xi)}{\partial\xi}X_0(\xi) + \varepsilon^2\frac{\partial F(t,\xi)}{\partial t} + \varepsilon^3, \ldots \qquad (19.51)$$

since

$$\frac{\partial \tilde{X}(t,\xi)}{\partial t} = X(t,\xi) - X_0(\xi).$$

Thus eq. (19.50) will be equal to eq. (19.51) with an accuracy to terms of the order of smallness of ε^3, if we select the available $P(\xi)$ and $F(t,\xi)$ in such a manner that the following relation is satisfied:

$$\frac{\partial F(t,\xi)}{\partial t} = \left(\tilde{X}\frac{\partial}{\partial\xi}\right)X(t,\xi) - \frac{\partial \tilde{X}(t,\xi)}{\partial\xi}X_0(\xi) - P(\xi). \qquad (19.52)$$

However, in view of the fact that

$$\tilde{X}(t,\xi) = \sum_{\nu\neq 0}\frac{e^{i\nu t}}{i\nu}X_\nu(\xi); \quad X(t,\xi) = \sum_{\nu} e^{i\nu t}X_\nu(\xi), \qquad (19.53)$$

we may write

$$\left(\tilde{X}\frac{\partial}{\partial\xi}\right)X(t,\xi) - \frac{\partial \tilde{X}(t,\xi)}{\partial\xi}X_0(\xi) = \sum_{\nu',\nu''(\nu'\neq 0)} e^{i(\nu'+\nu'')t}\frac{1}{i\nu'} \times \left(X_{\nu'}\frac{\partial}{\partial\xi}\right)X_{\nu''}(\xi) - \sum_{\nu\neq 0}\frac{e^{i\nu t}}{i\nu}\frac{\partial X_\nu(\xi)}{\partial\xi}X_0(\xi), \quad (19.54)$$

where, in the sum

$$\sum_{\substack{\nu', \nu'' \\ (\nu' \neq 0)}}$$

the summation is extended to all pairs (ν', ν'') of the frequencies ν figuring in the sums of eq. (19.53).

Equation (19.54) may, consequently, be represented by a sum of the form

$$\left(\tilde{X}\frac{\partial}{\partial \xi}\right) X(t, \xi) - \frac{\partial \tilde{X}(t, \xi)}{\partial \xi} X_0(\xi) = \sum_{\substack{\mu \\ (\mu = \nu, \nu' + \nu'')}} e^{i\mu t}\Phi_\mu(\xi),$$

and eq. (19.52) will be satisfied if we assume that

$$P(\xi) = \Phi_0(\xi) = \underset{t}{M}\left\{\left(\tilde{X}\frac{\partial}{\partial \xi}\right) X(t, \xi) - \frac{\partial \tilde{X}(t, \xi)}{\partial \xi} X_0(\xi)\right\} =$$
$$= \underset{t}{M}\left\{\left(\tilde{X}\frac{\partial}{\partial \xi}\right) X(t, \xi)\right\}$$

and

$$F(t, \xi) = \sum_{\mu \neq 0} \frac{e^{i\mu t}}{i\mu}\Phi_\mu(\xi) = \overbrace{\left(\tilde{X}\frac{\partial}{\partial \xi}\right) X(t, \xi)} - \frac{\partial \tilde{\tilde{X}}(t, \xi)}{\partial \xi} X_0(\xi). \quad (19.55)$$

To summarize, we can affirm that, for values of ξ determined from the equation

$$\frac{d\xi}{dt} = \varepsilon \underset{t}{M}\{X(t, \xi)\} + \varepsilon^2 \underset{t}{M}\left\{\left(\tilde{X}\frac{\partial}{\partial \xi}\right) X(t, \xi)\right\}, \quad (19.56)$$

the expression

$$x = \xi + \varepsilon\tilde{X}(t, \xi) + \varepsilon^2 \overbrace{\left(\tilde{X}\frac{\partial}{\partial \xi}\right) X(t, \xi)} - \varepsilon^2 \frac{\partial \tilde{\tilde{X}}(t, \xi)}{\partial \xi} X_0(\xi) \quad (19.57)$$

will satisfy eq. (19.17) with an accuracy to terms of the order of ε^3.

Next, we will show that, if eq. (19.57) so obtained is considered a formula of substitution of variables transforming the unknown x determined by the exact eq. (19.17) into the new unknown ξ, then it will satisfy an equation of the form

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$$\frac{d\xi}{dt} = \varepsilon \underset{t}{M}\{X(t, \xi)\} + \varepsilon^2 \underset{t}{M}\left\{\left(X \frac{\partial}{\partial \xi}\right) X(t, \xi)\right\} + \varepsilon^3 \ldots \tag{19.58}$$

For this purpose, let us differentiate eq.(19.57), using the notations of eq.(19.55) to shorten the formulas.

Then we obtain

$$\begin{aligned}\frac{dx}{dt} &= \frac{d\xi}{dt} + \varepsilon \frac{\partial \tilde{X}(t, \xi)}{\partial \xi} \frac{d\xi}{dt} + \\ &\quad + \varepsilon^2 \frac{\partial F(t, \xi)}{\partial \xi} \frac{d\xi}{dt} + \varepsilon \frac{\partial \tilde{X}(t, \xi)}{\partial t} + \varepsilon^2 \frac{\partial F(t, \xi)}{\partial t} = \\ &= \left(1 + \varepsilon \frac{\partial \tilde{X}}{\partial \xi} + \varepsilon^2 \frac{\partial F}{\partial \xi}\right) \frac{d\xi}{dt} + \varepsilon \frac{\partial \tilde{X}(t, \xi)}{\partial t} + \varepsilon^2 \frac{\partial F(t, \xi)}{\partial t},\end{aligned} \tag{19.59}$$

where 1 denotes the unit matrix.

However, the very definition of the integrating operator yields

$$\begin{aligned}&\varepsilon \frac{\partial \tilde{X}(t, \xi)}{\partial t} + \varepsilon^2 \frac{\partial F(t, \xi)}{\partial t} = \\ &\quad = \varepsilon X(t, \xi) - \varepsilon \underset{t}{M}\{X(t, \xi)\} + \varepsilon^2 \left(\tilde{X} \frac{\partial}{\partial \xi}\right) X(t, \xi) - \\ &\quad\quad - \varepsilon^2 \frac{\partial \tilde{X}(t, \xi)}{\partial \xi} X_0(\xi) - \varepsilon^2 \underset{t}{M}\left\{\left(\tilde{X} \frac{\partial}{\partial \xi}\right) X(t, \xi)\right\},\end{aligned}$$

and therefore it follows from eq.(19.59) that

$$\begin{aligned}\frac{dx}{dt} &= \left(1 + \varepsilon \frac{\partial \tilde{X}}{d\xi} + \varepsilon^2 \frac{\partial F}{\partial \xi}\right) \frac{d\xi}{dt} + \varepsilon X(t, \xi) + \\ &\quad + \varepsilon^2 \left(\tilde{X} \frac{\partial}{\partial \xi}\right) X(t, \xi) - \varepsilon^2 \frac{\partial \tilde{X}(t, \xi)}{\partial \xi} X_0(\xi) - \\ &\quad\quad - \varepsilon X_0(\xi) - \varepsilon^2 \underset{t}{M}\left\{\left(\tilde{X} \frac{\partial}{\partial \xi}\right) X(t, \xi)\right\}.\end{aligned}$$

By virtue of eq.(19.17), this expression must be equal to the following:

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$$\varepsilon X(t, x) = \varepsilon X(t, \xi + \varepsilon\widetilde{X} + \varepsilon^2 F) =$$
$$= \varepsilon X(t, \xi) + \varepsilon^2\left(\widetilde{X}\frac{\partial}{\partial\xi}\right)X(t, \xi) + \varepsilon^3\ldots$$

This shows that the variable ξ satisfies the equation

$$\frac{d\xi}{dt} = \left(1 + \varepsilon\frac{\partial\widetilde{X}}{\partial\xi} + \varepsilon^2\frac{\partial F}{\partial\xi}\right)^{-1}\left[\varepsilon X_0(\xi) + \right.$$
$$\left. + \varepsilon^2\frac{\partial\widetilde{X}(t,\xi)}{\partial\xi}X_0(\xi) + \varepsilon^2 \underset{t}{M}\left\{\left(\widetilde{X}\frac{\partial}{\partial\xi}\right)X(t, \xi)\right\} + \varepsilon^3\ldots\right]. \qquad (19.60)$$

However, it is obvious that

$$\left(1 + \varepsilon\frac{\partial\widetilde{X}}{\partial\xi} + \varepsilon^2\frac{\partial F}{\partial\xi}\right)^{-1} = 1 - \varepsilon\frac{\partial\widetilde{X}(t,\xi)}{\partial\xi} + \varepsilon^2\ldots,$$

so that eq. (19.60) may be presented in the form

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi) + \varepsilon^2 \underset{t}{M}\left\{\left(\widetilde{X}\frac{\partial}{\partial\xi}\right)X(t, \xi)\right\} + \varepsilon^3\ldots,$$

coinciding with eq. (19.58).

Thus, if ξ satisfies eq. (19.58) whose right side differs from the right side of eq. (19.56) by terms of the order of smallness of ε^3, then eq. (19.58) is the exact solution of eq. (19.17).

Thus, as our second approximation, let us take:

$$x = \xi + \varepsilon\widetilde{X}(t, \xi), \qquad (19.61)$$

where ε is determined by eq. (19.56). In other words for the second approximation we adopt the form of the refined first approximation in which ε satisfies an equation which is no longer of the first approximation, but already of the second.

We will call eq. (19.57), in which ξ is determined from eq. (19.56), the refined second approximation.

As we have seen, the refined second approximation satisfies eq. (19.17) with an

error of the order of smallness of ε^3.

All above statements can be directly generalized to equations of the type

$$\frac{dx}{dt} = \varepsilon X(t, x) + \varepsilon^2 Y(t, x), \tag{19.62}$$

which contain terms of the second order of smallness.

In this case, the equations of second approximation take the form

$$\frac{d\xi}{dt} = \varepsilon \underset{t}{M}\{X(t, \xi)\} + \varepsilon^2 \underset{t}{M}\{Y(t, \xi)\} + \varepsilon^2 \underset{t}{M}\left\{\left(\widetilde{X}\frac{\partial}{\partial \xi}\right) X(t, \xi)\right\}, \tag{19.63}$$

and the expression of second approximation will be

$$x = \xi + \varepsilon \widetilde{X}(t, \xi) \tag{19.64}$$

Finally, for the refined second approximation we find

$$x = \xi + \varepsilon \widetilde{X}(t, \xi) + \varepsilon^2 \widetilde{Y}(t, \xi) + \varepsilon^2 \overline{\left(\widetilde{X}\frac{\partial}{\partial \xi}\right) X(t, \xi)} - \varepsilon^2 \frac{\partial \widetilde{\widetilde{X}}}{\partial \xi} X_0(\xi). \tag{19.65}$$

We note now that

$$\begin{aligned} &\underset{t}{M}\{\varepsilon X(t, \xi + \varepsilon \widetilde{X}) + \varepsilon^2 Y(t, \xi + \varepsilon \widetilde{X})\} = \\ &\qquad = \underset{t}{M}\{\varepsilon X(t, \xi + \varepsilon \widetilde{X}) + \varepsilon^2 Y(t, \xi)\} + \varepsilon^3 \ldots = \\ &= \underset{t}{M}\left\{\varepsilon X(t, \xi) + \varepsilon^2 \left(\widetilde{X}\frac{\partial}{\partial \xi}\right) X(t, \xi) + \varepsilon^2 Y(t, \xi)\right\} + \varepsilon^3 \ldots . \end{aligned} \tag{19.66}$$

Therefore, since the terms of the order of smallness of ε^3 are disregarded in the equations of second approximation, eq. (19.63) may be written indifferently either in the form

$$\frac{d\xi}{dt} = \underset{t}{M}\{\varepsilon X(t, \xi + \varepsilon \widetilde{X}) + \varepsilon^2 Y(t, \xi)\} \tag{19.67}$$

or in the form

$$\frac{d\xi}{dt} = \underset{t}{M}\{\varepsilon X(t, \xi + \varepsilon \widetilde{X}) + \varepsilon^2 Y(t, \xi + \varepsilon \widetilde{X})\}. \tag{19.68}$$

This means that the equations of second approximation may be obtained directly

from the exact equations (19.62), if the term x in their right sides is substituted by the form of the refined first approximation (or, what is the same thing, the form of the second approximation) and the mean is taken over the explicitly contained time t, treating the variables ξ in the process of averaging as constants, while terms of the third order of smallness are rejected.

This principle of averaging may be formulated as follows: Equations of the second approximation are obtained by taking the mean of the exact equations (19.62) on both sides where the refined first approximation with respect to explicitly contained time has been substituted. Indeed, the equations of second approximation follow from the relations

$$\underset{t}{M}\left\{\frac{dx}{dt}\right\} = \underset{t}{M}\{\varepsilon X(t, x) + \varepsilon^2 Y(t, x)\} \qquad (19.69)$$

[where, in both sides $\xi + \varepsilon\tilde{X}(t, \xi)$ is substituted for x], while, during the process of averaging, $\frac{d\xi}{dt}$, ξ are treated as constants and terms of the order of smallness of ε^3 can be neglected. It may be worth mentioning that, with this interpretation of the operation $\underset{t}{M}$, we obviously will have

$$\underset{t}{M}\left\{\frac{dx}{dt}\right\} = \underset{t}{M}\left\{\frac{d\xi}{dt} + \varepsilon\frac{\partial\tilde{X}}{\partial\xi}\frac{d\xi}{dt} + \varepsilon\frac{\partial\tilde{X}}{\partial t}\right\} =$$
$$= \frac{d\xi}{dt} + \varepsilon\underset{t}{M}\left\{\frac{\partial\tilde{X}}{\partial\xi}\right\}\frac{d\xi}{dt} + \varepsilon\underset{t}{M}\left\{\frac{\partial\tilde{X}}{\partial t}\right\} = \frac{d\xi}{dt},$$

so that eq.(19.69) is transformed into eq.(19.68).

In conclusion, a few remarks on setting up the higher approximations will be made.

Let the general equation in the standard form be

$$\frac{dx}{dt} = \varepsilon X(t, x) + \varepsilon^2 X_1(t, x) + \ldots + \varepsilon^m X_{m-1}(t, x), \qquad (19.70)$$

where X_k (t, x) are certain trigonometric sums of the same type as X (t, x).

Then, in order to set up the m^{th} approximation, let us consider the expression

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$$x = \xi + \varepsilon F_1(t, \xi) + \ldots + \varepsilon^m F_m(t, \xi), \qquad (19.71)$$

in which $F_k(t, \xi)$ are sums of the form

$$\sum_{\mu \neq 0} e^{i\mu t} F_{k\mu}(\xi)$$

while the variable ξ will be a solution of the equation

$$\frac{d\xi}{dt} = \varepsilon P_1(\xi) + \varepsilon^2 P_2(\xi) + \ldots + \varepsilon^m P_m(\xi). \qquad (19.72)$$

On substituting eq.(19.71) in eq.(19.70) and equating the coefficients of the same powers of ε to the m^{th} order inclusive, we select $F_1, \ldots, F_m$, and $P_1, \ldots P_m$ in such a manner that eq.(19.71) will satisfy eq.(19.70) with an accuracy to terms of the order of smallness of ε^{m+1}.

In this case, we obtain

$$P_1(\xi) = \underset{t}{M}\{X(t, \xi)\};$$

$$P_2(\xi) = \underset{t}{M}\left\{\left(\tilde{X}\frac{\partial}{\partial\xi}\right)X(t, \xi) + X_1(t, \xi)\right\}; \ \ldots \ F_1(t, \xi) = \tilde{X}(t, \xi);$$

$$F_2(t, \xi) = \left(\overline{\tilde{X}\frac{\partial}{\partial\xi}\right)X(t, \xi)} - \frac{\partial\tilde{\tilde{X}}}{\partial\xi}\underset{t}{M}\{X(t, \xi)\} + \tilde{X}_1(t, \xi); \ \ldots$$

If, after having determined $F_1, \ldots, F_m$ and $P_1, \ldots, P_m$, we consider eq.(19.71) as a certain formula for substitution of variables, transforming the unknown x into the new unknown ξ, this will be determined by an equation of the form

$$\frac{d\xi}{dt} = \varepsilon P_1(\xi) + \varepsilon^2 P_2(\xi) + \ldots + \varepsilon^m P_m(\xi) + \varepsilon^{m+1} \ldots \qquad (19.73)$$

Thus if the variable ξ satisfies eq.(19.73), differing from eq.(19.72) by terms of the order of smallness of ε^{m+1}, then eq.(19.71) represents the exact solution for eq.(19.70).

For this reason, the expression

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$$x = \xi + \varepsilon F_1(t, \xi) + \ldots + \varepsilon^{m-1} F_{m-1}(t, \xi),$$

in which ξ is determined by the equation of the m^{th} approximation (19.72), may be adopted as the m^{th} approximation. For such a ξ, eq.(19.71) will yield a refined m^{th} approximation satisfying the exact equation (19.70) with an error of the order of ε^{m+1}. We note that if the form of the improved $(m-1)^{th}$ approximation is known, the equation of m^{th} approximation can be directly found from the exact equation (19.70) on substituting this form and on forming the mean by the aid of the operator M_t. On the whole, in the applications of the above theory of perturbations, we use mainly the first approximation and only occasionally the second. The higher approximations are rarely employed, because of the rapid increase in the complexity of their construction.

As an example illustrating this theory, let us consider the oscillations of a physical pendulum which is a solid body, freely rotatable in a certain vertical plane about its point of suspension. Let the point of suspension perform, in the vertical direction, sinusoidal oscillations of small amplitude a, at high frequency ω in such a way that*

$$\omega > \omega_0 \frac{l}{a}; \quad \frac{a}{l} \ll 1. \tag{19.74}$$

It will be found that the unstable upper position of the pendulum can be made stable.

To consider this interesting phenomenon, let us set up the equation of oscillation of a pendulum with a vibrating point of suspension. Considering the damping proportional to the velocity**, we have

* Where l is the reduced length of the pendulum and $\omega_0 = \sqrt{\frac{g}{l}}$ is the natural frequency of the small oscillations.

** In fact, the equation of oscillations of a pendulum with its point of suspension at rest will become, as generally known, (continued on next page)

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$$\frac{d^2\theta}{dt^2}+\lambda\frac{d\theta}{dt}+\frac{g-a\omega^2\sin\omega t}{l}\sin\theta=0, \qquad (19.75)$$

where θ is the angle of deflection measured from the lower position of equilibrium; $y = a \sin \omega t$ is the vertical displacement of the point of suspension; λ is the coefficient of damping. With respect to the magnitude of damping we assume that, at the fixed point of suspension, the motion of the pendulum at small deflections from the lower position of equilibrium has an oscillatory character. Then, as is generally known,

$$\frac{\lambda^2}{4}<\omega^2. \qquad (19.76)$$

To find the small parameter in eq.(19.75), it is expedient to introduce "dimensionless" time. More specifically, the time t measured in seconds is replaced by the time τ for which the unit of measurement will be related to 2π which is the period of oscillation of the point of suspension, i.e., $\frac{1}{\omega}$. We have, obviously,

$$\tau=\omega t;\quad \frac{d}{d\tau}=\frac{1}{\omega}\frac{d}{dt};\quad \frac{d^2}{d\tau^2}=\frac{1}{\omega^2}\frac{d^2}{dt^2},$$

so that eq.(19.75) yields

$$\frac{d^2\theta}{d\tau^2}+\frac{\lambda}{\omega}\frac{d\theta}{d\tau}+\left\{\frac{g}{l\omega^2}-\frac{a}{l}\sin\tau\right\}\sin\theta=0. \qquad (19.77)$$

Let us put, for brevity,

** (cont'd)

$$\frac{d^2\theta}{dt^2}+\lambda\frac{d\theta}{dt}+\frac{g}{l}\sin\theta=0, \qquad (\alpha)$$

However, from the point of view of the principle of relativity, the motion of a pendulum with a vertically vibrating point of suspension is the equivalent to the motion of a pendulum with a point of suspension at rest in a field of "gravity" with an acceleration $g+y''$. On replacing g in eq.(α) by $g+y''$, we arrive at eq.(19.75).

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$$k = \frac{\omega_0}{\omega} : \frac{a}{l}; \quad \alpha = \frac{\lambda}{2\omega_0} k. \tag{19.78}$$

Then,

$$\frac{g}{l\omega^2} = \left(\frac{\omega_0}{\omega}\right)^2 = k^2 \left(\frac{a}{l}\right)^2; \quad \frac{\lambda}{\omega} = \frac{\lambda}{\omega_0}\frac{\omega_0}{\omega} = \frac{\lambda}{\omega_0} k \frac{a}{l} = 2\alpha \frac{a}{l},$$

and eq.(19.77) may be written in the form

$$\frac{d^2\theta}{d\tau^2} + 2\alpha \frac{a}{l}\frac{d\theta}{d\tau} + \left\{k^2\left(\frac{a}{l}\right)^2 - \frac{a}{l}\sin\tau\right\}\sin\theta = 0.$$

Taking the ratio of the amplitude of oscillation of the suspension point to the reduced length of the pendulum as our small parameter ε, we have finally

$$\frac{d^2\theta}{d\tau^2} + 2\varepsilon\alpha \frac{d\theta}{d\tau} + \{k^2\varepsilon^2 - \varepsilon\sin\tau\}\sin\theta = 0, \tag{19.79}$$

where, according to eqs.(19.74), (19.75), and (19.78), the constants α and k will be less than unity

$$\alpha < 1, \quad k < 1.$$

Since the equation so obtained, containing the small parameter ε, is not an equation of standard form, it must first be transformed into this form before direct application of the above theory becomes possible.

It turns out that, by means of a simple substitution of variables, this differential equation of the second order can be transformed into two equations of the first order in the standard form. For this purpose, we replace one unknown function of time θ by two new unknowns φ and Ω, using the formulas

$$\theta = \varphi - \varepsilon\sin\tau\sin\varphi, \tag{19.80}$$

$$\frac{d\theta}{d\tau} = \varepsilon\Omega - \varepsilon\cos\tau\sin\varphi. \tag{19.81}$$

Differentiating eq.(19.80) and comparing it with eq.(19.81), we have

$$\frac{d\theta}{d\tau}=\frac{d\varphi}{d\tau}-\varepsilon\sin\tau\cos\varphi\frac{d\varphi}{d\tau}-\varepsilon\cos\tau\sin\varphi=$$
$$=\varepsilon\Omega-\varepsilon\cos\tau\sin\varphi,$$

whence

$$(1-\varepsilon\sin\tau\cos\varphi)\frac{d\varphi}{dt}=\varepsilon\Omega. \qquad (19.82)$$

Differentiating eq.(19.81) and substituting in eq.(19.79), we get

$$\frac{d^2\theta}{d\tau^2}=\varepsilon\frac{d\Omega}{d\tau}-\varepsilon\cos\tau\cos\varphi\frac{d\varphi}{d\tau}+\varepsilon\sin\tau\sin\varphi=$$
$$=(\varepsilon\sin\tau-k^2\varepsilon^2)\sin\theta-2\gamma\varepsilon\frac{d\theta}{d\tau},$$

so that

$$\varepsilon\frac{d\Omega}{d\tau}=\varepsilon\cos\tau\cos\varphi\frac{d\varphi}{d\tau}+$$
$$+\varepsilon\sin\tau\{\sin(\varphi-\varepsilon\sin\tau\sin\varphi)-\sin\varphi\}-$$
$$-k^2\varepsilon^2\sin(\varphi-\varepsilon\sin\tau\sin\varphi)+2\gamma\varepsilon(\varepsilon\Omega-\varepsilon\cos\tau\sin\varphi),$$

whence, eliminating ε and taking eq.(19.82) into consideration, we obtain

$$\frac{d\Omega}{d\tau}=\{\sin(\varphi-\varepsilon\sin\tau\sin\varphi)-\sin\varphi\}\sin\tau-$$
$$-k^2\varepsilon\sin(\varphi-\varepsilon\sin\tau\sin\varphi)+\frac{\varepsilon\Omega\cos\tau\cos\varphi}{1-\varepsilon\sin\tau\cos\varphi}-$$
$$-2\gamma\varepsilon(\Omega-\cos\tau\sin\varphi). \qquad (19.83)$$

This indicates that, because of eqs.(19.82) and (19.83), the variables φ, Ω satisfy the differential equations in the standard form

$$\left.\begin{aligned}&\frac{d\varphi}{dt}=\varepsilon\Omega+\varepsilon^2\ldots,\\&\frac{d\Omega}{d\tau}=\varepsilon\{-\sin^2\tau\sin\varphi\cos\varphi-k^2\sin\varphi+\end{aligned}\right\} \qquad (19.84)$$

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$$+\Omega\cos\tau\cos\varphi - 2\alpha\Omega + 2\alpha\cos\tau\sin\varphi\} + \varepsilon^2\ldots \Big]$$

Applying the method of the mean to these expressions and taking account of the identical relations

$$\underset{\tau}{M}\{\cos\tau\} = 0, \quad \underset{\tau}{M}\{\sin^2\tau\} = \frac{1}{2},$$

we obtain the equations of first approximation in the form

$$\left.\begin{aligned} \frac{d\varphi}{d\tau} &= \varepsilon\Omega, \\ \frac{d\Omega}{d\tau} &= -\varepsilon\left\{\frac{1}{2}\sin\varphi\cos\varphi + k^2\sin\varphi + 2\gamma\Omega\right\}. \end{aligned}\right\} \quad (19.85)$$

These two equations of the first order (19.85), are obviously equivalent to the single equation of the second order

$$\frac{d^2\varphi}{d\tau^2} + 2\varepsilon\alpha\frac{d\varphi}{d\tau} + \varepsilon^2\left(k^2 + \frac{1}{2}\cos\varphi\right)\sin\varphi = 0. \quad (19.86)$$

The resultant equation of the first order is far simpler than the exact eq.(19.79), already by virtue of the fact that it does not contain the time explicitly. This equation constitutes the oscillation equation for a system similar to the pendulum with a fixed suspension point in which the "restoring force" is proportional not to $\sin\varphi$ but to $\left(k^2 + \frac{1}{2}\cos\varphi\right)\sin\varphi$. It is interesting to note, among other things, that for example certain gyroscopes (Bibl.54) are systems of this kind.

In the absence of damping ($\alpha = 0$), eq.(19.86) is completely solved in elliptic functions. However, for considering the question in which we are interested, we do not need expressions of the general solution. Equation (19.86) indicates directly that this equation admits of the quasi-static solution $\varphi = \pi$, corresponding to the upper position of equilibrium of the pendulum.

For studying the stability, consider the small deflections $\delta\varphi = \varphi - \pi$ from this

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position. Then, the equation of variation for $\delta\varphi$ takes the form

$$\frac{d^2\delta\varphi}{d\tau^2}+2\varepsilon\alpha\frac{d\,\delta\varphi}{d\tau}+\varepsilon^2\left(\frac{1}{2}-k^2\right)\delta\varphi=0. \qquad (19.87)$$

Since here $\varepsilon\alpha > 0$, the condition of stability will be

$$\frac{1}{2}-k^2>0,$$

i.e., bearing in mind the definition of k:

$$\omega>\sqrt{2}\,\omega_0\frac{l}{a}. \qquad (19.88)$$

Thus, if the frequency of vibration of the point of suspension is great enough to satisfy the inequality (19.88), then the upper position of the pendulum becomes stable.

For instance let l = 40 cm, a = 2 cm. In this case, the condition (19.88) will yield

$$\omega>\sqrt{2}\sqrt{\frac{981}{40}}\,20\cong 140\ \frac{1}{sec}.$$

The upper position of the pendulum will consequently be stable if the number of cycles of the oscillation of the suspension point is more than $\frac{\omega}{2\pi}$, i.e., more than 22.3 cps.

If we consider analogously the quasi-static solution φ = 0 corresponding to the lower position of equilibrium, it will become obvious that this will remain stable for any values of k and that the frequency of oscillations at small deflections, without allowing for damping, will be equal to $\varepsilon\sqrt{\frac{1}{2}+k^2}$ for the time τ and correspondingly to

$$\varepsilon\omega\sqrt{\frac{1}{2}+k^2}=\sqrt{\frac{1}{2}\left(\frac{a\omega}{l}\right)^2+\omega_0^2}$$

for the time t.

For the above concrete example, at a frequency of oscillation of the suspension

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point equal to 60 cps $\left(\omega = 377 \frac{1}{\sec}\right)$, the frequency of the small oscillations of the pendulum will be $\omega_\mu = 14.2 \frac{1}{\sec}$, while in the case of a suspension point at rest, this frequency will be equal to $\omega_v = 4.94 \frac{1}{\sec}$. The effective restoring force is here increased by a factor of $\left(\frac{\omega_\mu}{\omega_v}\right)^2 = 8.2$. This force, at small deflections, will thus be the same as in a corresponding ordinary pendulum which is 8.2 times as heavy.

We note finally that the equation of first approximation (19.86) permits a consideration of the question of stability not only at small but also at large deflections.

Next, we will discuss the oscillations of a pendulum in second approximation. It is obvious that the equations of second approximation coincide with the equations of first approximations.

For this reason, in constructing the second approximation, another possible type of motion of the pendulum will be investigated. It has been found that a pendulum can rotate synchronously at the angular velocity ω, expending work for overcoming the resistances, provided that these resistances do not exceed a certain quantity. Here oscillations of the pendulum are possible about an axis rotating uniformly at an angular velocity exactly equal to ω.

In order to simplify the calculations slightly we exclude the action of gravity, assuming that the motion of the pendulum takes place in a horizontal plane. Then, putting $k = 0$ in eq.(19.79), we obtain

$$\frac{d^2\theta}{d\tau^2} + 2\varepsilon\alpha \frac{d\theta}{d\tau} - \varepsilon \sin\tau \sin\theta = 0. \qquad (19.89$$

The angle θ measures the deflection of the axis of the pendulum from a certain fixed axis and, since we intend to study the oscillations of a pendulum about an axis rotating at constant angular velocity ω, it is expedient to replace the angle θ

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by the angle ψ:

$$\psi = \theta - \omega t$$

or for the dimensionless time τ, used in eq. (19.89),

$$\psi = \theta - \tau.$$

For the angle ψ the equation of oscillation will obviously be

$$\frac{d^2\psi}{d\tau^2} + 2\varepsilon\alpha\frac{d\psi}{d\tau} - \varepsilon\sin\tau\sin(\psi+\tau) + 2\varepsilon\alpha = 0. \qquad (19.90)$$

To reduce this eq. (19.90) to the standard form, let

$$\psi = \psi, \quad \frac{d\psi}{d\tau} = \sqrt{\varepsilon}\,v. \qquad (19.91)$$

As a result, we obtain two equations of the first order with respect to the unknowns ψ and v:

$$\frac{d\psi}{d\tau} = \sqrt{\varepsilon}\,v,$$

$$\frac{dv}{d\tau} = \sqrt{\varepsilon}\sin\tau\sin(\psi+\tau) - 2\sqrt{\varepsilon}\,\alpha - 2(\sqrt{\varepsilon})^2\alpha v, \qquad (19.92)$$

in which $\sqrt{\varepsilon}$ may be taken as the small parameter.

Since

$$\sin\tau\sin(\psi+\tau) = \frac{1}{2}\cos\psi - \frac{1}{2}\cos(\psi+2\tau),$$

the refined first approximation (second approximation) will be

$$\psi = \psi, \quad v = \Omega - \frac{\sqrt{\varepsilon}}{2}\widetilde{\cos}(\psi+2\tau) = \Omega - \frac{\sqrt{\varepsilon}}{4}\sin(\psi+2\tau). \qquad (19.93)$$

Substituting eq. (19.93) in the right sides of eq. (19.92) and taking the mean with respect to τ with constant ψ, Ω we arrive at the equations of second approximation:

$$\left.\begin{aligned} \frac{d\psi}{d\tau} &= \sqrt{\varepsilon}\,\Omega; \\ \frac{d\Omega}{d\tau} &= \frac{\sqrt{\varepsilon}}{2}\cos\psi - 2\sqrt{\varepsilon}\,\alpha - 2\varepsilon\alpha\Omega, \end{aligned}\right\} \qquad (19.94)$$

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or

$$\frac{d^2\psi}{d\tau^2} + 2\varepsilon\alpha \frac{d\psi}{d\tau} - \frac{\varepsilon \cos\psi}{2} + 2\varepsilon\chi = 0.$$

If we return to the time t measured in seconds $\left(t = \frac{\tau}{\omega}\right)$, then the resultant equation of second approximation can be represented in the form

$$\frac{d^2\psi}{dt^2} + \lambda \frac{d\psi}{dt} - \frac{a\omega^2}{2l} \cos\psi + \lambda\omega = 0. \tag{19.95}$$

We note, among other things, that, in the notation adopted, the equation of first approximation would be

$$\frac{d^2\psi}{dt^2} - \frac{a\omega^2}{2l} \cos\psi + \lambda\omega = 0. \tag{19.96}$$

It differs from the equation of second approximation by the absence of the term $\lambda \frac{d\psi}{dt}$, due to the damping.

On considering the equation of second approximation, we see that it admits the quasi-static solutions

$$\psi = \psi_0, \quad \text{where} \quad \frac{a\omega^2}{2l} \cos\psi_0 = \lambda\omega, \tag{19.97}$$

corresponding to the rotation of the pendulum ($\theta = \omega t + \psi_0$) at constant angular velocity ω, provided only that

$$\lambda\omega < \frac{a\omega^2}{2l}. \tag{19.98}$$

At

$$\lambda\omega > \frac{a\omega^2}{2l} \tag{19.99}$$

such quasi-static solutions are impossible.

To investigate the stability of the quasi-static solutions (19.97) in the case of eq. (19.98), let us consider the small deflections ψ from ψ_0:

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$$\psi = \psi_0 + \delta\psi.$$

For small deflections, eq.(19.95) gives

$$\frac{d^2\delta\psi}{dt^2} + \lambda \frac{d\delta\psi}{dt} + \frac{a\omega^2}{2l} \sin \psi_0 \, \delta\psi = 0. \tag{19.100}$$

Investigating the corresponding characteristic equation

$$p^2 + \lambda p + \frac{a\omega^2}{2l} \sin \psi_0 = 0, \tag{19.101}$$

it becomes obvious that, in view of the fact that the coefficient λ, for $\frac{a\omega^2}{2l} \sin \psi_0 > 0$ is positive, the real parts of the roots of this equation are negative; at $\frac{a\omega^2}{2l} \sin \psi_0 < 0$, this equation has a root with a positive real part.

Thus the solution (19.97) is stable for $\sin \psi_0 > 0$ and unstable for $\sin \psi_0 < 0$. We have, consequently, one stable quasi-static solution $0 < \psi_0 < \pi$ and one unstable $\pi < \psi_0 < 2\pi$.

We note that, if we confined ourselves to the consideration of the equation of first approximation (19.96), then eq.(19.100) would not contain the term $\lambda \frac{d\delta\psi}{dt}$, and the characteristic equation would have the form

$$p^2 + \frac{a\omega^2}{2l} \sin \psi_0 = 0.$$

Consequently, for $\sin \psi_0 > 0$, its roots are purely imaginary, with a real part equal to zero, and the question of stability remains unsettled. The possibility of such cases was mentioned above. As we see, in considering the second approximation, the real part of the roots of the characteristic equation differs from zero, so that the question of stability can be clarified.

We will say a few words in conclusion on the subject of the condition for the existence of the quasi-static solutions (19.97).

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We note that if I denotes the moment of inertia of the pendulum, then $I\lambda\omega$ obviously represents the moment of the forces of resistance for a pendulum rotating at an angular velocity ω.

By multiplying the moment of the forces of resistance by ω, we obtain the power N consumed in overcoming these forces

$$N = I\lambda\omega^2.$$

The condition (19.98) shows that for a steady rotation of the pendulum at an angular velocity ω to be possible, the power expended in overcoming the forces of resistance must not reach a certain limiting value, which is

$$N < \frac{I}{2}\frac{a}{l}\omega^3. \qquad (19.102)$$

Thus, for example, if the moment of inertia of the pendulum is $I = 0.5$ kg-cm-sec^2, the reduced length is $l = 40$ cm, and the point of suspension executes 60 oscillations per second $\left(\omega = 377\,\frac{1}{\text{sec}}\right)$ at an amplitude of $a = 2$ cm, then

$$\frac{I}{2}\frac{a}{l}\omega^3 = \frac{(377)^3}{80}\ Kg\ cm\ sec^{-1} = 6698\ Kg\ m\ sec^{-1}.$$

In this case, according to condition (19.102), for the pendulum to be able to rotate at an angular velocity ω (60 rps) the power expended on overcoming the resistances must not exceed 6698 kg-m-sec^{-1}.

Section 20. The Case of a Rapidly Rotating Phase

In this Section, a generalization of the method of the mean will be given for the case of a system with a rapidly rotating phase.

Respective studies were made by D.N. Zubarev in collaboration with one of the authors of the present monograph (Bibl.9).

Consider the dynamic system whose state is characterized by the angular variable α, and by r variables $x_1, x_2, \ldots, x_r$, and is described by the following system of equations:

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$$\left.\begin{aligned} \frac{dx_k}{dt} &= X_k(\alpha,\ x_1,\ \ldots,\ x_r) \quad (k = 1,\ 2,\ \ldots,\ r), \\ \frac{d\alpha}{dt} &= \lambda\omega(x_1,\ \ldots,\ x_r) + A(\alpha,\ x_1,\ \ldots,\ x_r), \end{aligned}\right\} \qquad (20.1)$$

where λ is a large parameter; λ_ω corresponds to the frequency of rotation α; X_k $(\alpha,\ x_1,\ \ldots,\ x_r)$, A $(\alpha,\ x_1,\ \ldots,\ x_r)$ are periodic functions of the angular variable α with a period of 2π.

We note that, in the special case where ω = const, while A $(\alpha,\ x_1,\ \ldots,\ x_r) = 0$, the system (20.1) may be directly reduced to the standard form.

In fact, in this case we have

$$\frac{dx_k}{dt} = X_k(\lambda\omega t + \varphi,\ x_1,\ \ldots,\ x_r) \quad (\varphi = \text{const}),$$

whence, introducing the new independent variable

$$\lambda\omega t = \tau,$$

we obtain equations of the type of eq.(19.14), where

$$\varepsilon = \frac{1}{\lambda\omega}.$$

In the general case of the system (20.1), the fundamental idea of the method of the mean can be used.

It will be demonstrated below that the variable α can be eliminated from the right sides of eq.(20.1) with any degree of accuracy in the expansion in power series of $\frac{1}{\lambda}$. For this purpose, we define the substitution of variables

$$\left.\begin{aligned} x_k &= \bar{x}_k + \sum_{n=1}^{\infty} \frac{1}{\lambda^n} \xi_k^{(n)}(\bar{\alpha},\ \bar{x}_1,\ \ldots,\ \bar{x}_r) \\ &\qquad\qquad (k = 1,\ 2,\ \ldots,\ r), \\ \alpha &= \bar{\alpha} + \sum_{n=1}^{\infty} \frac{1}{\lambda^n} U_n(\bar{\alpha},\ \bar{x}_1,\ \ldots,\ \bar{x}_r), \end{aligned}\right\} \qquad (20.2)$$

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by means of which the system (20.1) can be reduced to the form

$$\left.\begin{aligned} \frac{d\bar{x}_k}{dt} &= \sum_{n=0}^{\infty} \frac{1}{\lambda^n} X_k^{(n)}(\bar{x}_1, \ldots, \bar{x}_r), \\ \frac{d\bar{\alpha}}{dt} &= \lambda\omega(\bar{x}_1, \ldots, \bar{x}_r) + \sum_{n=0}^{\infty} \frac{1}{\lambda^n} \Omega_n(\bar{x}_1, \ldots, \bar{x}_r) \end{aligned}\right\} \tag{20.3}$$

in such a way that the coefficients in eq.(20.3) no longer depend on the angular variable $\bar{\alpha}$.

The physical meaning of the transformation (20.2) consists in a resolution of the actual motion described by the variables $x_1, x_2, \ldots, x_r, \alpha$ into a mean motion with the coordinates $\bar{x}_1, \ldots, \bar{x}_r$ and a "vibration", described by the angle $\bar{\alpha}$ and the functions

$$U_n(\bar{\alpha}, \bar{x}_1, \ldots, \bar{x}_r) \text{ and } \xi_k(\bar{\alpha}, \bar{x}_1, \ldots, \bar{x}_r).$$

The determination of the functions entering into eq.(20.2), speaking generally, is not single-valued, because of the arbitrariness with which the various terms of the expansion can be related either to the main or the higher terms of the series. This fact has been noted repeatedly.

In the case where we have a certain concrete expansion (20.2), we may always perform the substitution of variables of the form

$$\bar{x}_k = \bar{\bar{x}}_k + \varepsilon f_k(\bar{\bar{x}}_1, \ldots, \bar{\bar{x}}_r) + \varepsilon^2 \ldots,$$

resulting in another possible form of the expansion, since $\bar{\bar{x}}_k$, with the same justification, can be adopted as the new $\bar{x}_k$.

To obtain definite single-valued expressions for the coefficients of eq.(20.2), certain additional conditions must be established. Assume that U_n and $\xi_k^{(n)}$ must not contain null harmonics with respect to $\bar{\alpha}$, which would mean that $\bar{x}_k$ and $\bar{\alpha}$ include all of the mean motion.

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Other conditions might also be imposed. For example, if the system is canonical, it could be required that the equations of the mean motion (20.3) should likewise canonical. Since the fact that they are not single-valued is of a trivial character, the possible cases will not be discussed here.

On substituting eq.(22.2) in eq.(20.1) and equating the terms in λ, λ^{1}, λ^{-1}, we obtain a system of four equations for determining the six functions Ω_0, Ω_1, $X_k^{(0)}$, $X_k^{(1)}$, $\xi_k^{(1)}$, U_1.

$$\left.\begin{aligned}
&X_k^{(0)} + \frac{\partial \xi_k^{(1)}}{\partial \alpha}\,\omega = X_k \qquad (k = 1,\ 2,\ \ldots,\ r),\\
&X_k^{(1)} + \frac{\partial \xi_k^{(2)}}{\partial \alpha}\,\omega + \frac{\partial \xi_k^{(1)}}{\partial \alpha}\,\Omega_0 + \sum_{q=1}^{r} \frac{\partial \xi_k^{(1)}}{\partial x_q} X_q^{(0)} =\\
&\qquad\qquad = \frac{\partial X_k}{\partial \alpha} U_1 + \sum_{q=1}^{r} \frac{\partial X_k}{\partial x_q}\xi_q^{(1)},\\
&\Omega_1 + \frac{\partial U_2}{\partial \alpha}\,\omega + \frac{\partial U_1}{\partial \alpha}\,\Omega_0 + \sum_{q=1}^{r} \frac{\partial U_1}{\partial x_q} X_q^{(0)} =\\
&\qquad = \sum_{p=1}^{r} \frac{\partial \omega}{\partial x_p}\xi_p^{(2)} + \frac{1}{2}\sum_{p,q} \frac{\partial^2 \omega}{\partial x_p\,\partial x_q}\xi_p^{(1)}\xi_q^{(1)} +\\
&\qquad\qquad + \frac{\partial A}{\partial \alpha} U_1 + \sum_{q=1}^{r} \frac{\partial A}{\partial x_q}\xi_q^{(1)},\\
&\Omega_0 + \frac{\partial U_1}{\partial \alpha}\,\omega = \sum_{q=1}^{r} \frac{\partial \omega}{\partial x_q}\xi_q^{(1)} + A.
\end{aligned}\right\} \qquad (20.4)$$

In the system (20.4) the number of unknowns is larger than the number of equations, which is in complete agreement with the above remark that they are not single-valued. The missing equations are obtained from the condition of the absence of null harmonics in $\xi_k^{(n)}$ and U_n, i.e.,

$$\widetilde{\xi}_k^{(N)} = 0, \quad \widetilde{U}_n = 0 \tag{20.5}$$

(the wavy bar denotes averaging with respect to $\bar{\alpha}$).

Let us expand the function $A(\bar{\alpha}, \bar{x}_1, \ldots, \bar{x}_r)$, $X_k(\bar{\alpha}, \bar{x}_1, \ldots, \bar{x}_r)$ into Fourier series:

$$\left.\begin{aligned} A(\bar{\alpha}, \bar{x}_1, \ldots, \bar{x}_r) &= \sum_{-\infty < m < \infty} A_m e^{im\bar{\alpha}}, \\ X_k(\bar{\alpha}, \bar{x}_1, \ldots, \bar{x}_r) &= \sum_{-\infty < m < \infty} X_{km} e^{im\bar{\alpha}}. \end{aligned}\right\} \tag{20.6}$$

Averaging eq.(20.4) with respect to $\bar{\alpha}$, we find

$$X_k^{(0)} = X_{k0}, \tag{20.7}$$

$$\xi_k^{(0)} = \frac{1}{\omega} \sum_{n \neq 0} X_{km} \frac{e^{in\bar{\alpha}}}{in}, \tag{20.8}$$

$$\Omega_0 = A_0, \tag{20.9}$$

$$U_1 = \sum_{n \neq 0} \frac{A_n e^{in\bar{\alpha}}}{in\omega} - \sum_{n \neq 0} \sum_{q=1}^{r} \frac{\partial \omega}{\partial \bar{x}_q} \frac{X_{q,n}}{n^2\omega^2} e^{in\bar{\alpha}}, \tag{20.10}$$

$$\Omega_1 = \frac{1}{2} \sum_{p,q} \frac{\partial^2 \omega}{\partial \bar{x}_p \partial \bar{x}_q} \widetilde{\xi_p^{(1)} \xi_q^{(1)}} + \widetilde{\frac{\partial A}{\partial \bar{\alpha}} U_1} + \sum_{q=1}^{r} \widetilde{\frac{\partial A}{\partial \bar{x}_q} \xi_q^{(1)}}, \tag{20.11}$$

$$X_k^{(1)} = \widetilde{\frac{\partial X_k}{\partial \bar{\alpha}} U_1} + \sum_{q=1}^{r} \widetilde{\frac{\partial X_k}{\partial \bar{x}_q} \xi_q^{(1)}}. \tag{20.12}$$

With the use of eqs.(20.8) and (20.10), we reduce eqs.(20.11) and (20.12) to the form:

$$\Omega_1 = \frac{1}{2} \sum_{\substack{p,q,n \\ (n \neq 0)}} \frac{\partial^2 \omega}{\partial \bar{x}_p \partial \bar{x}_q} \frac{1}{\omega^2 n^2} X_{p,n} X_{q,-n} + \sum_{\substack{q,n \\ (n \neq 0)}} \frac{\partial \omega}{\partial \bar{x}_q} \frac{1}{i\omega^2 n} A_n X_{q,-n} -$$

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$$-\sum_{n \neq} \frac{1}{\omega} A_n A_{-n} - \sum_{\substack{q, n \\ (n \neq 0)}} \frac{1}{i\omega n} \frac{\partial A_n}{\partial \bar{x}_q} X_{q, -n} \qquad (20.13)$$

$$X_k^{(1)} = -\sum_{(n \neq 0)} \frac{1}{\omega} X_{kn} A_{-n} - \sum_{\substack{n, q \\ (n \neq 0)}} \frac{1}{in\omega} \frac{\partial X_{kn}}{\partial \bar{x}_q} X_{q, -n} + \sum_{\substack{n, q \\ (n \neq 0)}} \frac{\partial \omega}{\partial \bar{x}_q} \frac{1}{i\omega^2 n} X_{k, n} X_{q, -n} \qquad (20.14)$$

The expressions (20.7)-(20.10), (20.13) and (20.14) yield the required solution of the system (20.4).

Let us pass from the complex Fourier series (20.6) to the real series

$$\left.\begin{aligned} A &= A_0 + \sum_{n=1}^{\infty} \{f_n \cos n\bar{\alpha} + g_n \sin n\bar{\alpha}\}, \\ X_k &= X_{k0} + \sum_{n=1}^{\infty} \{F_{kn} \cos n\bar{\alpha} + G_{kn} \sin n\bar{\alpha}\}. \end{aligned}\right\} \qquad (20.15)$$

Let us represent the formulas for the substitution of variables (20.2), by means of eqs.(20.8), (20.10), and (20.15), in the form

$$\left.\begin{aligned} x_k &= \bar{x}_k + \frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{1}{n\omega} \{-G_{kn} \cos n\bar{\alpha} + F_{kn} \sin n\bar{\alpha}\} + O\left(\frac{1}{\lambda^2}\right), \\ \alpha &= \bar{\alpha} + \frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{1}{n\omega} \{-g_n \cos n\bar{\alpha} + f_n \sin n\bar{\alpha}\} - \\ &- \frac{1}{\lambda} \sum_{\substack{n, q \\ (n \neq 0)}} \frac{1}{n^2\omega^2} \frac{\partial \omega}{\partial \bar{x}_q} \{F_{q,n} \cos n\bar{\alpha} + G_{q,n} \sin n\bar{\alpha}\} + O\left(\frac{1}{\lambda^2}\right). \end{aligned}\right\} \qquad (20.16)$$

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The system of equations (20.3), after substituting the coefficients from eqs.(20.7)-(20.10), (20.13), and (20.14), takes the form

$$\frac{d\bar{x}_k}{dt} = X_{k0} - \frac{1}{\lambda}\sum_{n=1}^{\infty}\frac{1}{2\omega}\{F_{k,n} f_n + G_{k,n} g_n\} - $$

$$- \frac{1}{\lambda}\sum_{\substack{n,q\\(n\neq 0)}}\frac{1}{2n\omega}\left\{\frac{\partial F_{k,n}}{\partial \bar{x}_q} G_{q,n} - \frac{\partial G_{k,n}}{\partial \bar{x}_q} F_{q,n}\right\} +$$

$$+ \frac{1}{\lambda}\sum_{\substack{q,n\\(n\neq 0)}}\frac{1}{2\omega^2 n}\frac{\partial\omega}{\partial \bar{x}_q}\{F_{k,n} G_{q,n} - G_{k,n} F_{q,n}\} + O\left(\frac{1}{\lambda^2}\right). \qquad (20.17)$$

$$\frac{d\bar{\alpha}}{dt} = \lambda\omega + A_0 + \frac{1}{\lambda}\sum_{\substack{n,p,q\\(n\neq 0)}}\frac{1}{4\omega^2 n^2}\frac{\partial^2\omega}{\partial \bar{x}_p \partial \bar{x}_q}\{F_{pn} F_{qn} + G_{pn} G_{qn}\} -$$

$$- \frac{1}{\lambda}\sum_{\substack{n,q\\(n\neq 0)}}\frac{1}{2\omega^2 n}\frac{\partial\omega}{\partial \bar{x}_q}\{g_n F_{q,n} - f_n G_{q,n}\} + \frac{1}{\lambda}\sum_{\substack{n,q\\(n\neq 0)}}\frac{1}{2\omega n}\left\{\frac{\partial g_n}{\partial \bar{x}_q} F_{q,n} - \frac{\partial f_n}{\partial \bar{x}_q} G_{q,n}\right\} -$$

$$- \frac{1}{\lambda}\sum_{n}\frac{1}{2\omega}\{f_n^2 + g_n^2\} + O\left(\frac{1}{\lambda^2}\right). \qquad (20.18)$$

The system of equations (20.17), (20.18) yields the solution of the problem posed at the beginning of this Section, with an accuracy to terms of the first order of smallness inclusive, with respect to the parameter $\frac{1}{\lambda}$. The first group of equations of this system (20.7) expresses the systematic motion. Equation (20.18) for $\bar{\alpha}$ expresses the "vibration". Thus the systematic motion is separated from the "vibration" with an accuracy to terms of the order of $\frac{1}{\lambda^2}$.

Let us consider, as an example, the motion of a charged particle in a magnetic field. This problem is of interest for many questions of theoretical physics.

For example, in cosmic electrodynamics, the problem of investigating the paths of cosmic particles in nonuniform fields arises. This same problem also arises in the theory of certain electrotechnical instruments used in radio engineering, analogous to the magnetron. The exact integration of the equations of motion of a

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charged particle in non-uniform electric and magnetic fields is difficult and, in most cases, can be performed only by numerical methods. Even this is not always feasible. In particular, the difficulty of numerical calculations become almost insurmountable as soon as the particle executes a large number of revolutions along a Larmor circle during the time of its motion. It is precisely in this case that the above-described method of asymptotic approximation can be utilized, preventing any calculation difficulties.

Assume that the magnetic field varies only slightly over the length of the Larmor radius:

$$R_L \frac{1}{H} \frac{dH}{dx} \ll 1, \tag{20.19}$$

where $R_L = \frac{v}{\omega_H}$ is the radius of the Larmor circle, $\omega_H = \frac{eH}{mc}$ the Larmor frequency and v the velocity of the particle in a plane perpendicular to the magnetic field.

Then the charged particle moves mainly in a spiral around a magnetic line of force, rotating at about the distance of the Larmor radius and "drifting" in a direction perpendicular to the magnetic field. By making use of this fact, we can construct simplified mean equations for the motion of the center of gravity of the Larmor circle.

The satisfaction of the condition (20.19) is favored by high values and uniformity of the magnetic field and by small values of the velocity of the particle. The condition (20.19), however, can also be satisfied at high particle velocity if the field is sufficiently large and uniform, and also at small magnetic fields if the velocity of the particle is sufficiently small and the field is sufficiently uniform.

In this example we will investigate the motion of the charged particle in a non-uniform electric and magnetic field, assuming that the magnetic field varies little over the length of the Larmor radius.

The equations of motion of a charged particle in a magnetic and electric field,

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in nonrelativistic approximation, are of the form

$$\left.\begin{aligned} \frac{d\boldsymbol{v}}{dt} &= \boldsymbol{F} + \frac{e}{mc}[\boldsymbol{v}\boldsymbol{H}], \quad \boldsymbol{F} = \frac{e\boldsymbol{E}}{m}, \\ \frac{d\boldsymbol{r}}{dt} &= \boldsymbol{v}. \end{aligned}\right\} \qquad (20.20)$$

Let us select a curvilinear system of coordinates with the orthogonal coordinates τ_o, τ_1, τ_2 in the direction of the lines of the magnetic field and the two perpendiculars to it:

$$\boldsymbol{\tau}_0 = \frac{\boldsymbol{H}}{H}, \quad \boldsymbol{\tau}_1 = [\boldsymbol{\tau}_2\boldsymbol{\tau}_0], \quad \boldsymbol{\tau}_2 = [\boldsymbol{\tau}_0\boldsymbol{\tau}_1], \quad \boldsymbol{\tau}_0 = [\boldsymbol{\tau}_1\boldsymbol{\tau}_2]. \qquad (20.21)$$

We then write eq. (20.20) in the form

$$\left.\begin{aligned} \frac{d\boldsymbol{v}}{dt} &= \boldsymbol{F} + \omega_H [\boldsymbol{v}\boldsymbol{\tau}_0], \\ \frac{d\boldsymbol{r}}{dt} &= \boldsymbol{v}, \end{aligned}\right\} \qquad (20.22)$$

where $\omega_H = \frac{eH(r)}{mc}$ is the Larmor frequency.

Let us represent eq. (20.22) in such a form that the rotation of the particle at an angular velocity of ω_H will be explicitly separated. For this purpose, let us resolve the vector of velocity of the particle v with respect to the orthogonal coordinates τ_o, τ_1, τ_2:

$$\left.\begin{gathered} \boldsymbol{v} = u\boldsymbol{\tau}_0 + w\{\boldsymbol{\tau}_1 \cos\alpha + \boldsymbol{\tau}_2 \sin\alpha\} \\ (v^2 = w^2 + u^2), \end{gathered}\right\} \qquad (20.23)$$

where u is the velocity component parallel to the field and w the component perpendicular to it.

According to eq. (20.23), eq. (20.22) assumes the form

$$\frac{du}{dt}\boldsymbol{\tau}_0 + u\frac{d\boldsymbol{\tau}_0}{dt} + \frac{dw}{dt}\{\boldsymbol{\tau}_1 \cos\alpha + \boldsymbol{\tau}_2 \sin\alpha\} +$$

$$+ w\left\{\frac{d\boldsymbol{\tau}_1}{dt}\cos\alpha + \frac{d\boldsymbol{\tau}_2}{dt}\sin\alpha\right\} + w\{-\boldsymbol{\tau}_1 \sin\alpha + \boldsymbol{\tau}_2 \cos\alpha\}\frac{d\alpha}{dt} =$$

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$$= F + \omega_H w \{\tau_1 \sin\alpha - \tau_2 \cos\alpha\}. \qquad (20.24)$$

By multiplying eq.(20.24) successively by τ_0, $\tau_1 \cos\alpha + \tau_2 \sin\alpha$, and $\tau_2 \cos\alpha - \tau_1 \sin\alpha$, we obtain the following equations for $\frac{du}{dt}$, $\frac{dw}{dt}$, $\frac{d\alpha}{dt}$:

$$\left.\begin{aligned} \frac{du}{dt} &= (F\tau_0) - w\left\{\tau_0 \frac{d\tau_1}{dt}\cos\alpha + \tau_0 \frac{d\tau_2}{dt}\sin\alpha\right\}, \\ \frac{dw}{dt} &= (F\tau_1)\cos\alpha + (F\tau_2)\sin\alpha - u(\tau_1\cos\alpha + \tau_2\sin\alpha)\frac{d\tau_0}{dt}, \\ w\frac{d\alpha}{dt} &= -\omega_H w + F\{\tau_2\cos\alpha - \tau_1\sin\alpha\} - \\ &\quad - \{\tau_2\cos\alpha - \tau_1\sin\alpha\}\left\{u\frac{d\tau_0}{dt} + w\left(\frac{d\tau_1}{dt}\cos\alpha + \frac{d\tau_2}{dt}\sin\alpha\right)\right\}. \end{aligned}\right\} \qquad (20.25)$$

We have

$$\frac{d\tau_i}{dt} = \frac{\partial\tau_i}{\partial t} + (v\nabla)\tau_i = \frac{\partial\tau_i}{\partial t} + u(\tau_0\nabla)\tau_i + $$
$$+ w\{(\tau_1\nabla)\tau_i\cos\alpha + (\tau_2\nabla)\tau_i\sin\alpha\} \quad (i = 0,\ 1,\ 2) \qquad (20.26)$$

and

$$\tau_0(\tau_0\nabla) + \tau_1(\tau_1\nabla) + \tau_2(\tau_2\nabla) = \nabla. \qquad (20.27)$$

Put $\frac{\partial\tau_i}{\partial\tau} = 0$, i.e., consider that the magnetic field does not depend on time, although this limitation could also be omitted. Then, with the aid of eqs.(20.26) and (20.27), eq.(20.25) takes the form

$$\frac{du}{dt} = (F\tau_0) + \frac{w^2}{2}\operatorname{div}\tau_0 + uw\{\tau_1(\tau_0\nabla)\tau_0\cos\alpha +$$
$$+ \tau_2(\tau_0\nabla)\tau_0\sin\alpha\} + \frac{w^2}{2}\{\tau_1(\tau_1\nabla)\tau_0 - \tau_2(\tau_2\nabla)\tau_0\}\cos 2\alpha +$$
$$+ \frac{w^2}{2}\{\tau_1(\tau_2\nabla)\tau_0 + \tau_2(\tau_1\nabla)\tau_0\}\sin 2\alpha, \qquad (20.28)$$

$$\frac{dw}{dt} = -\frac{uw}{2}\operatorname{div}\tau_0 + \{(F\tau_1) - u^2\tau_1(\tau_0\nabla)\tau_0\}\cos\alpha +$$
$$+ \{(F\tau_2) - u^2\tau_2(\tau_0\nabla)\tau_0\}\sin\alpha -$$
$$- \frac{uw}{2}\{\tau_1(\tau_1\nabla)\tau_0 - \tau_2(\tau_2\nabla)\tau_0\}\cos 2\alpha -$$
$$- \frac{uw}{2}\{\tau_1(\tau_2\nabla)\tau_0 + \tau_2(\tau_1\nabla)\tau_0\}\sin 2\alpha, \qquad (20.29)$$

$$\frac{d\alpha}{dt} = -\omega_H - \frac{u}{2}\{\tau_2(\tau_1\nabla)\tau_0 - \tau_1(\tau_2\nabla)\tau_0 + 2\tau_2(\tau_0\nabla)\tau_1\} +$$
$$+ \frac{1}{w}\{(F\tau_2) - u^2\tau_2(\tau_0\nabla)\tau_0 + w^2\tau_1(\tau_1\nabla)\tau_2\}\cos\alpha +$$
$$+ \frac{1}{w}\{-(F\tau_1) + u^2\tau_1(\tau_0\nabla)\tau_0 - w^2\tau_2(\tau_2\nabla)\tau_1\}\sin\alpha -$$
$$- \frac{u}{2}\{\tau_1(\tau_2\nabla)\tau_0 + \tau_2(\tau_1\nabla)\tau_0\}\cos 2\alpha +$$
$$+ \frac{u}{2}\{\tau_1(\tau_1\nabla)\tau_0 - \tau_2(\tau_2\nabla)\tau_0\}\sin 2\alpha. \qquad (20.30)$$

To these equations we must still add the second equation of the system (20.22):

$$\frac{dr}{dt} = u\tau_0 + w\{\tau_1\cos\alpha + \tau_2\sin\alpha\}. \qquad (20.31)$$

The equations of motion of a charged particle in a nonuniform field in the form of eqs.(20.28)-(20.31) are convenient for application of the above method of asymptotic approximation to the case of a magnetic field differing little from a uniform field and satisfying the conditions (20.19).

Let us perform in the system (20.28)-(20.31) a substitution of variables analogous to the substitution of eq.(20.16):

$$r = \bar{r} + \frac{w}{\omega_H}(\tau_2\cos\bar{\alpha} - \tau_1\sin\bar{\alpha}),$$
$$\alpha = \bar{\alpha} + \frac{1}{\omega_H}(g_1\cos\bar{\alpha} - f_1\sin\bar{\alpha}) +$$

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\left.\begin{aligned}
&+\frac{w}{\omega_H^2}(\tau_1\cos\bar{\alpha}+\tau_2\sin\bar{\alpha})\nabla\omega_H+\\
&+\frac{1}{2\omega_H}(g_2\cos 2\bar{\alpha}-f_2\sin 2\bar{\alpha}),\\
u&=\bar{u}-\frac{1}{\omega_H}\sum_{n=1,2}\frac{1}{n}\{-G_{4n}\cos n\bar{\alpha}+F_{4n}\sin n\bar{\alpha}\},\\
w&=\bar{w}-\frac{1}{\omega_H}\sum_{n=1,2}\frac{1}{n}\{-G_{5n}\cos n\bar{\alpha}+F_{5n}\sin n\bar{\alpha}\},
\end{aligned}\right\}\quad(20.32)
$$

where u and w correspond to the variables x_4, x_5 of the system (20.1) and $\omega_H = -\lambda\omega$; f_n, g_n, F_{4n}, G_{4n}, F_{5n}, G_{5n} are the coefficients of the corresponding harmonics in eqs.(20.28)-(20.31).

$$
\left.\begin{aligned}
f_1&=\frac{1}{w}\{(F,\tau_2)-u^2\tau_2(\tau_0\nabla)\tau_0+w^2\tau_1(\tau_1\nabla)\tau_2\},\\
g_1&=\frac{1}{w}\{-(F,\tau_1)+u^2\tau_1(\tau_0\nabla)\tau_0-w^2\tau_2(\tau_2\nabla)\tau_1\},\\
f_2&=-\frac{u}{2}\{\tau_1(\tau_2\nabla)\tau_0+\tau_2(\tau_1\nabla)\tau_0\},\\
g_2&=\frac{u}{2}\{\tau_1(\tau_1\nabla)\tau_0-\tau_2(\tau_2\nabla)\tau_0\},\\
F_{41}&=uw\tau_1(\tau_0\nabla)\tau_0,\quad F_{42}=\frac{w^2}{2}\{\tau_1(\tau_1\nabla)\tau_0-\tau_2(\tau_2\nabla)\tau_0\},\\
F_{51}&=(\tau_1F)-u^2\tau_1(\tau_0\nabla)\tau_0,\\
F_{52}&=-\frac{uw}{2}\{\tau_1(\tau_1\nabla)\tau_0-\tau_2(\tau_2\nabla)\tau_0\},\\
G_{41}&=uw\tau_2(\tau_0\nabla)\tau_0,\\
G_{42}&=\frac{w^2}{2}\{\tau_1(\tau_2\nabla)\tau_0+\tau_2(\tau_1\nabla)\tau_0\},\\
G_{51}&=(\tau_2F)-u^2\tau_2(\tau_0\nabla)\tau_0,\\
G_{52}&=-\frac{uw}{2}\{\tau_1(\tau_2)\nabla)\tau_0+\tau_2(\tau_1\nabla)\tau_0\}.
\end{aligned}\right\}\quad(20.33)
$$

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The first formula of the system (20.32) expresses the rotation of the particle around the Larmor circle about a mean position, while the second, third, and fourth equations describe the influence of the nonuniformity of the field and of the external force on the angle of rotation α and the velocity components u and w.

As a result of the transformation (20.32), eqs.(20.28)-(20.31) will no longer contain the angular variable α. In our subsequent discussion we will everywhere omit the symbols of averaging for the variables, writing $\bar{r}$, $\bar{u}$, $\bar{w}$, as simply r, u, w, which cannot cause confusion, since we will only deal with averaged variables in the following.

We shall perform all the calculations with an accuracy to terms proportional to $\frac{1}{\omega_H}$. Then, in the approximate equations for $\frac{du}{dt}$ and $\frac{dw}{dt}$, as will be seen in the sequel, it is sufficient to retain only the terms of zero order with respect to $\frac{1}{\omega_H}$.

In the 0^{th} approximation, eqs.(20.28) and (20.29) yield:

$$\left.\begin{aligned} \frac{du}{dt} &= (F\tau_0) + \frac{w^2}{2}\operatorname{div}\tau_0, \\ \frac{dw}{dt} &= -\frac{uw}{2}\operatorname{div}\tau_0, \end{aligned}\right\} \qquad (20.34)$$

since

$$\left.\begin{aligned} X_{10} &= (F\tau_0) + \frac{w^2}{2}\operatorname{div}\tau_0, \\ X_{50} &= -\frac{uw}{2}\operatorname{div}\tau_0. \end{aligned}\right\} \qquad (20.35)$$

From the system (20.34) follows the law of conservation of energy for the averaged motion. In fact, when multiplying the first equation of the system (20.24) by u, the second by w, and forming the sum, we obtain

$$u\frac{du}{dt} + w\frac{dw}{dt} = u(F\tau_0) = -\frac{1}{m}\frac{dV}{dt},$$

where V is the potential energy of the particle.

Consequently,

$$\frac{1}{2}(u^2+w^2)+\frac{V}{m}=\text{const}. \tag{20.36}$$

Let us find the adiabatic invariant from the second equation of the system (20.34). We have

$$\frac{dw}{dt}=-\frac{uw}{2}\left(H\nabla\left(\frac{1}{H}\right)\right)=-\frac{Hw}{2}\left(u\nabla\left(\frac{1}{H}\right)\right)=$$
$$=-\frac{Hw}{2}\left(\frac{dr}{dt}\nabla\left(\frac{1}{H}\right)\right)=-\frac{Hw}{2}\frac{d}{dt}\left(\frac{1}{H}\right), \tag{20.37}$$

since div $H = 0$ and $u\tau_0 \approx \frac{dr}{dt}$.

As a result of integrating eq. (20.37), we get

$$\frac{w^2}{H}=\text{const}. \tag{20.38}$$

Thus the quantity $\frac{w^2}{H}$ is an adiabatic invariant, i.e., is not preserved exactly but only with an accuracy to terms of the order of $\frac{1}{\omega_H}$.

We might have supplemented eq. (20.38) with the higher terms in $\frac{1}{\omega_H}$, beginning with the first one, and obtained an explicit expression for the approximate integral of motion, preserving its value to any degree of accuracy assigned in advance.

The physical meaning of the adiabatic invariance of the quantity $\frac{w^2}{H}$ is that the magnetic flux across the Larmor circle is a quantity constant within an accuracy to terms of the order of smallness of $\frac{1}{\omega_H}$.

As a result of the transformation (20.32), eq. (20.31) takes the form

$$\frac{dr}{dt}=u\tau_0+\frac{w}{2\omega_H}(\tau_1 f_1+\tau_2 g_1)+$$
$$+\frac{w^2}{2\omega_H}((\tau_2\nabla)\tau_1-(\tau_1\nabla)\tau_2)+\frac{1}{2\omega_H}(\tau_1 G_{31}\quad\tau_2 F_{31})-$$
$$-\frac{w^2}{2\omega_H^2}(\tau_1(\tau_2\nabla\omega_H)-\tau_2(\tau_1\nabla\omega_H))+O\left(\frac{1}{\omega_H^2}\right). \tag{20.39}$$

allowing for

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$$
\begin{aligned}
w(\tau_1 f_1 + \tau_2 g_1) + \tau_1 G_{51} - \tau_2 F_{51} = \\
= \tau_1 \{2(\tau_2 F) - 2u^2 \tau_2 (\tau_0 \nabla) \tau_0 + w^2 \tau_1 (\tau_1 \nabla) \tau_2\} + \\
+ \tau_2 \{-2(\tau_1 F) + 2u^2 \tau_1 (\tau_0 \nabla) \tau_0 - w^2 \tau_2 (\tau_2 \nabla) \tau_1\}.
\end{aligned}
$$

Let us rewrite eq.(20.39) in the form

$$
\begin{aligned}
\frac{dr}{dt} = u\tau_0 + \frac{1}{\omega_H} \{\tau_1 (F\tau_2) - \tau_2 (F\tau_1)\} - \\
- \frac{u^2}{\omega_H} \{\tau_1 \tau_2 (\tau_0 \nabla) \tau_0 - \tau_2 \tau_1 (\tau_0 \nabla) \tau_0\} + \\
+ \frac{w^2}{2\omega_H} \{\tau_1 \tau_1 (\tau_1 \nabla) \tau_2 - \tau_2 \tau_2 (\tau_2 \nabla) \tau_1\} + \\
+ \frac{w^2}{2\omega_H} \{(\tau_2 \nabla) \tau_1 - (\tau_1 \nabla) \tau_2\} - \\
- \frac{w^2}{2\omega_H} \{\tau_1 (\tau_2 \nabla \omega_H) - \tau_2 (\tau_1 \nabla \omega_H)\} + O\left(\frac{1}{\omega_H^2}\right) \qquad (20.40)
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{dr}{dt} = \tau_0 \left\{u + \frac{w^2}{2\omega_H} (\tau_0 \operatorname{rot} \tau_0)\right\} + \\
+ \tau_0 \times \left\{- \frac{1}{\omega_H} F + \frac{w^2}{2\omega_H^2} \nabla \omega_H + \frac{u^2}{\omega_H} (\tau_0 \nabla) \tau_0\right\}, \qquad (20.41)
\end{aligned}
$$

since

$$
\begin{aligned}
\tau_1 (F\tau_2) - \tau_2 (F\tau_1) = [F \times \tau_0], \\
\tau_1 \tau_2 (\tau_0 \nabla) \tau_0 - \tau_2 \tau_1 (\tau_0 \nabla) \tau_0 = [(\tau_0 \nabla) \tau_0 \times \tau_0], \\
(\tau_2 \nabla) \tau_1 - (\tau_1 \nabla) \tau_2 = \tau_0 (\tau_0 \operatorname{rot} \tau_0) - \tau_1 \tau_1 (\tau_1 \nabla) \tau_2 + \tau_2 \tau_2 (\tau_2 \nabla) \tau_1.
\end{aligned}
$$

In eq.(20.41), the small correction

$$
\frac{w^2}{2\omega} (\tau_0 \operatorname{rot} \tau_0) \tau_0
$$

to the main longitudinal term $u\tau_0$ may be neglected.

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It is obvious that in eq. (20.41), in the approximation adopted, the terms perpendicular to the field H begin with terms of the order of $\frac{1}{\omega_H}$, while those parallel to the field H are determined only with an accuracy to $\frac{1}{\omega_H}$. Maintaining the same accuracy, quantities of the type $\tau_0 \frac{c}{\omega_H}$ may be added to this equation. Indeed, if from our r we proceed to another substitution (cf. remark on cases which are not "single-valued"):

$$r = r_{new} + \frac{\tau_0}{\omega_H^2} f(\alpha, r), \qquad (20.42)$$

shifting r along the line of the magnetic field by a quantity of the order $\frac{1}{\omega_H^2}$, which is negligible at the degree of accuracy adopted, then on differentiation of eq. (20.42), the terms $\frac{\tau_0}{\omega_H} f'_\alpha$ are added in addition to the insubstantial terms of the order of $\frac{1}{\omega_H^2}$.

Let us make use of this arbitrariness, and determine u, for example, so that it will exactly equal the component of velocity r of the center of the Larmor circle parallel to the magnetic field. Then the new u will be equal to the old u plus $\frac{L}{\omega_H}$. In eq. (20.41) we may simply substitute u new for u old, since the difference between these quantities in order of magnitude is less than the terms retained in eq. (20.41).

Thus finally we obtain the following system of equations determining the motion of the center of the Larmor circle:

$$\left.\begin{aligned} \frac{du}{dt} &= (F\tau_0) + \frac{w^2}{2}\,\mathrm{div}\,\tau_0, \\ \frac{dw}{dt} &= -\frac{uw}{2}\,\mathrm{div}\,\tau_0, \\ \frac{dr}{dt} &= \tau_0 u + \tau_0 \times \left\{ -\frac{1}{\omega_H} F + \frac{w^2}{2\omega_H^2} \nabla \omega_H + \frac{u^2}{\omega_H} (\tau_0 \nabla) \tau_0 \right\}. \end{aligned}\right| \qquad (20.43)$$

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It is not hard to see the physical meaning of the different terms in eq. (20.43): $\tau_0 u$ is the component of the velocity vector of the particle directed along the magnetic field;

$$\frac{1}{\omega_H}[F\tau_0] = \frac{c}{H^2}[EH] \quad (E \perp H)$$

is the velocity of drift of the particle under the action of the electric and magnetic fields;

$$\frac{w^2}{2\omega_H^2}[\tau_0 \times \operatorname{grad}\omega_H] = \frac{mcw^2}{2eH^3}[H \times \nabla H] -$$

is the velocity of the drift due to the nonuniformity of the magnetic field;

$$\frac{u^2}{\omega_H}[(\tau_0\nabla)\tau_0 \times \tau_0] = -\frac{u^2}{\omega_H R}[n\tau], \quad |n| = 1$$

(where R is the radius of curvature of the lines of the magnetic field and n is the principal normal to the lines of the magnetic field) is the velocity of drift due to the curvature of the lines of the magnetic field, or the velocity of the "centrifugal" drift.

The last equation of system (20.43) may also be written in the form

$$\frac{dr}{dt} = u\frac{\omega_H}{\omega_H} + \frac{v^2+u^2}{2\omega_H^3}[\omega_H \times \nabla\omega_H] + \frac{1}{\omega_H^2}[F\omega_H] + \frac{u^2}{\omega_H^2}\left\{\operatorname{rot}\omega_H - \frac{\omega_H}{\omega_H^2}(\omega_H \operatorname{rot}\omega_H)\right\}, \quad (20.44)$$

$$v^2 = u^2 + w^2,$$

where $\omega_H = \omega_H\tau_0$ has been used and the identity $(\tau_0\nabla)\tau_0 = -[\tau_0 \operatorname{rot}\tau_0]$ has been taken into account, or in the form

$$\frac{dr}{dt} = u\frac{H}{H} + \frac{c}{H^2}[EH] + \frac{mc(v^2+u^2)}{2eH^3}[H\nabla H] + \frac{mcu^2}{eH^2}\left\{\operatorname{rot}H - \frac{H}{H^2}(H\operatorname{rot}H)\right\}. \quad (20.45)$$

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In the special case where rot H = 0, eq.(20.45) takes the form

$$\frac{dr}{dt} = u\frac{H}{H} + \frac{c}{H^2}[EH] + \frac{mc^2(v^2 - u^2)}{2eH^3}[H\nabla H] \qquad (20.46)$$

Equation (20.43) describes the motion of the center of the Larmor circle with an accuracy to $\frac{1}{\omega_H^2}$.

We note in conclusion that the above general asymptotic method of the mean with a rapidly rotating phase may also be employed for the investigation of gyroscopic systems.

CHAPTER V

JUSTIFICATION OF THE ASYMPTOTIC METHODS

Section 21. Justification of the Method of the Mean

The problem of justification of the asymptotic methods may be investigated from various points of view.

We may, for example seek the conditions under which the difference between the exact solution and its asymptotic approximation for sufficiently small values of the parameter becomes as small as desired over a time interval as long as desired, but which is still finite.

We may also pose problems considerably more complex, attempting to establish the correspondence between those properties of the exact and approximate solutions that depend on their behavior over an infinite interval.

In the present Section we will consider the first of these problems, since it is the simpler one.

Since the above-described asymptotic methods allow a reduction to the method of the mean, the problem, for the sake of generality, will be formulated with respect to its application to a system of differential equations in the standard form.

Thus, we shall consider the system of equations

$$\frac{dx}{dt} = \varepsilon X(t,x) \tag{21.1}$$

(x, X are points of an n-dimensional Euclidean space) with the small parameter ε.

Let us construct for it the corresponding system of averaged equations

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$$\frac{d\xi}{dt} = \varepsilon X_0(\xi) \tag{21.2}$$

and let us proceed to the proof of a theorem establishing that, under very general conditions, the difference x (t) − ξ (t) may be made as small as desired, for a sufficiently small value of ε, over as long an interval $0 < t < T$ as may be desired. Since ξ (t) depends on t by way of the product εt, it follows that, for ξ to be able to depart significantly from its original value during the course of this time interval, i.e. for this time interval to be sufficiently long from the point of view of the variation of ξ, a quantity of the order $\frac{L}{\varepsilon}$ must be taken for T, where L may be made as large as desired for a sufficiently small ε.

Let us therefore formulate the assertion as to the smallness of the error involved x (t) − ξ (t) of the first approximation in the following way:

Theorem. If the function x (t, x) satisfies the conditions:

a) For a certain domain D, positive constants M and λ can be indicated such that, for all real values $t \geq 0$ and for any points x, x', x'' of this domain, the inequality

$$|X(t,x)| \leq M; \ |X(t,x') - X(t,x'')| \leqslant \lambda |x' - x''|. \tag{21.3}$$

will be satisfied.

b) Uniformly with respect to x, the domain D contains a limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X(t,x)\,dt = X_0(x). \tag{21.4}$$

Then to any positive ρ, η as small as desired, and to any L as large as desired, a positive ε_0 may be associated such that, if $\xi = \xi$ (t) is the solution of the equation

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi),$$

which is determinate in $0 < t < \infty$ and lies in the mentioned domain D together with

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its ρ-neighborhood*, then, for $0 < \varepsilon < \varepsilon_0$ in the intervals $0 < t < \frac{L}{\varepsilon}$ the following inequality will hold:

$$|x(t) - \xi(t)| < \eta,$$

in which $x = x(t)$ represents the solution of the equation

$$\frac{dx}{dt} = \varepsilon X(t, x),$$

coinciding with $\xi(t)$ for $t = 0$.

Proof. Let us fix a certain positive number a and construct the function

$$\Delta_a(x) = \begin{cases} A_a \left\{1 - \frac{|x|^2}{a^2}\right\}^2, & |x| \leqslant a, \\ 0, & |x| > a, \end{cases} \tag{21.5}$$

where the positive constant A_a is defined by the relation:

$$\int\limits_n \Delta_a(x)\, dx = 1, \tag{21.6}$$

in which the integration is performed over the whole space E_n under consideration; dx denotes an infinitesimal element of an ordinary n-dimensional Euclidean volume.

Obviously, the function $\Delta_a(x)$ so introduced is bounded, together with its partial derivatives, to the second order inclusive. Since this function and its derivatives vanish identically, it is obvious, for $|x| > a$, that the integral

$$I_a = \int\limits_{E_n} \left| \frac{\partial \Delta_a(x)}{\partial x} \right| dx \tag{21.7}$$

is finite for all positive values for a.

Noting this, consider the function

$$u(t, x) = \int\limits_D \Delta_a(x - x') \left\{ \int\limits_0^t [X(t, x') - X_0(x')]\, dt \right\} dx'. \tag{21.8}$$

* We shall denote as the ρ-neighborhood of a certain set A, the set of all points whose distance to A is less than ρ.

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By virtue of condition b) we may construct a monotonously decreasing function f (t) tending toward zero as $t \to \infty$, such that in the entire domain D

$$\left|\frac{1}{t}\int_0^t [X(t,x) - X_0(x)]\,dt\right| < f(t). \qquad (21.9)$$

We therefore have

$$|u(t,x)| \leqslant tf(t)\int_D \Delta_a(x-x')\,dx' \leqslant tf(t)\int_{E_n} \Delta_a(x-x')\,dx' =$$
$$= tf(t)\int_{E_n} \Delta_a(x')\,dx',$$

i.e.,

$$|u(t,x)| \leqslant tf(t). \qquad (21.10)$$

We have, further,

$$\left|\frac{\partial u(t,x)}{\partial x}\right| \leqslant tf(t)\int_D \left|\frac{\partial \Delta_a(x-x')}{\partial x}\right| dx' \leqslant tf(t)\int_{E_n}\left|\frac{\partial \Delta_a(x)}{\partial x}\right| dx,$$

or, in view of eq. (21.7),

$$\left|\frac{\partial u(t,x)}{\partial x}\right| \leqslant l_a t f(t). \qquad (21.11)$$

On the other hand, owing to condition a),

$$|X_0(x)| \leqslant M; \quad |X_0(x') - X_0(x'')| \leqslant$$
$$\leqslant \lambda |x'-x''|; \quad x,\ x',\ x'' \in D, \qquad (21.12)$$

and therefore

$$|X(t,x') - X_0(x') - X(t,x) + X_0(x)| \leqslant$$
$$\leqslant 2\lambda |x'-x|; \quad x,\ x' \in D. \qquad (21.13)$$

We note now, from eq. (21.8), that

$$\frac{\partial u(t,x)}{\partial t} = \int_D \{X(t,x') - X_0(x')\}\,\Delta_a(x-x')\,dx',$$

whence, on the basis of eq.(21.13), we can prove that, in the domain D, the inequality

$$\left|\frac{\partial u(t,x)}{\partial t} - \left\{X(t,x) - X_0(x)\right\}\int\limits_D \Delta_a(x-x')\,dx'\right| \leqslant 2\lambda a. \qquad (21.14)$$

is valid. However, by definition of the function $\Delta_a(x)$, for any point x, the a-neighborhood of which belongs to D, we have:

$$\int\limits_D \Delta_a(x-x')\,dx' = \int\limits_{|x-x'|<a} \Delta_a(x-x')\,dx' = 1$$

In this way, the relation (21.14) for these points gives

$$\left|\frac{\partial u(t,x)}{\partial t} - X(t,x) - X_0(x)\right| \leqslant 2\lambda a. \qquad (21.15)$$

Let us now fix the number a such that

$$a < \rho; \quad a < \frac{\eta^*}{8\lambda L e^{L\lambda}}; \quad \text{where } \eta^* = \min(\eta, \rho), \qquad (21.16)$$

and let us introduce the functions

$$F(\varepsilon) = \sup_{|\tau| \leqslant L}\left|\tau f\left(\frac{\tau}{\varepsilon}\right)\right|; \qquad \Phi(t) = \frac{1}{t^2}\int\limits_0^t t f(t)\,dt.$$

We have obviously

$$F(\varepsilon) \to 0, \quad \varepsilon \to 0; \qquad \Phi(t) \to 0, \quad t \to \infty.$$

We may therefore find a positive ε_0 so small that for any positive ε not exceeding ε_0, the inequalities

$$F(\varepsilon) < a; \quad F(\varepsilon) < \frac{\eta^*}{2}; \quad \Phi\left(\frac{L}{\varepsilon}\right) \leqslant \frac{\eta^*}{4L^2 e^{L\lambda}(\lambda + l_a M)}. \qquad (21.17)$$

will be satisfied.

After making such a selection, consider the expression

$$\bar{x} = \bar{x}(t) = \xi(t) + \varepsilon u(t, \xi(t)), \qquad (21.18)$$

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where ξ (t) is the solution of eq.(21.2) belonging, together with its ρ-neighborhood, to the domain D. From eqs.(21.16), and (21.17), we have

$$|\varepsilon u(t,\ \xi)| \leqslant \varepsilon t f(t) \leqslant F(\varepsilon) < a < \rho \qquad (21.19)$$

in the interval

$$0 < t < \frac{L}{\varepsilon}, \qquad (21.20)$$

and therefore, in this interval, $\bar{x}$ (t) $\in$ D.

We have, further,

$$\frac{d\bar{x}}{dt} = \varepsilon X(t,\ \bar{x}) = R, \qquad (21.21)$$

where

$$R = \frac{d\xi}{dt} + \varepsilon \frac{\partial u}{\partial \xi}\frac{d\xi}{dt} + \varepsilon \frac{\partial u}{\partial t} - \varepsilon X(t,\ \xi + \varepsilon u) =$$
$$= \varepsilon \left\{\frac{\partial u}{\partial t} - X(t,\ \xi) - X_0(\xi)\right\} + \varepsilon^2 \frac{\partial u}{\partial \xi} X_0(\xi) +$$
$$+ \varepsilon \{X(t,\ \xi) - X(t,\ \xi + \varepsilon u)\}.$$

Whence, in consequence of the inequalities (21.10), (21.11), 21.12), and (21.15), we obtain

$$|R| \leqslant 2\lambda a\varepsilon + I_a M \varepsilon^2 t f(t) + \lambda \varepsilon^2 t f(t)$$

In this way, we find, in the interval of eq.(21.20) under consideration,

$$\int_0^t e^{\varepsilon\lambda(t-\tau)} |R(\tau)|\, d\tau \leqslant e^{L\lambda} \int_0^{L/\varepsilon} |R(t)|\, dt <$$
$$< \left\{2\lambda a L + (I_a M + \lambda) L^2 \Phi\left(\frac{L}{\varepsilon}\right)\right\} e^{\lambda L},$$

or, in view of eqs.(21.16), (21.17),

$$\int_0^t e^{\varepsilon\lambda(t-\tau)} |R(\tau)|\, d\tau < \frac{\eta^*}{4} + \frac{\eta^*}{4} = \frac{\eta^*}{2}.$$

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so that

$$\int_0^t e^{\varepsilon\lambda(t-\tau)}|R(\tau)|d\tau < \frac{\eta}{2}; \qquad \int_0^t e^{\varepsilon\lambda(t-\tau)}|R(\tau)|d\tau < \frac{\rho}{2}. \tag{21.22}$$

Let now $x = x(t)$ represent a solution of eq.(21.1) for which $x(0) = \xi(0)$.

Then, in the interval

$$0 < t < t^*; \quad t^* \leqslant \frac{L}{\varepsilon}, \tag{21.23}$$

in which $x(t) \in D$, we may write:

$$|X(t,x) - X(t,\bar{x})| \leqslant \lambda|x - \bar{x}|,$$

whence, eq.(21.21) will yield

$$\left|\frac{d(x-\bar{x})}{dt}\right| \leqslant \lambda\varepsilon|x-\bar{x}| + |R(t)|.$$

Since the difference $x - \bar{x}$ vanishes at $t = 0$, it follows that

$$|x - \bar{x}| \leqslant \int_0^t e^{\varepsilon\lambda(t-\tau)}|R(\tau)|d\tau.$$

On the basis of eq.(21.22), therefore, we see that in the interval (21.23) the following inequalities hold:

$$|x - \bar{x}| < \frac{\eta}{2}; \quad |x - \bar{x}| < \frac{\rho}{2},$$

from which, in consequence of eqs.(21.18) and (21.19), we obtain

$$|x - \xi| < \frac{\eta}{2} + F(\varepsilon) < \eta; \quad |x - \xi| < \frac{\rho}{2} + F(\varepsilon) < \rho. \tag{21.24}$$

It will be shown below that the term t^* may be taken equal to $\frac{L}{\varepsilon}$.

If this cannot be done, then the inequality

$$d > |\xi - x| \tag{21.25}$$

cannot hold over the entire interval $\left(0, \frac{L}{\varepsilon}\right)$, since in the latter case we would have $x(t) \in D$ for each value of t from the interval $\left(0, \frac{L}{\varepsilon}\right)$. But since the

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inequality (21.25) is known to hold for sufficiently small values of t, it is clear from considerations of continuity that there must exist a t_1 such that, in the interval $(0, t_1)$, this inequality is valid and that, in addition,

$$|x(t_1) - \xi(t_1)| > \rho - \delta, \tag{21.26}$$

where any number, as low as desired, may be taken for δ. Let us take

$$\delta = \frac{1}{2}\left\{\frac{\rho}{2} - F(\varepsilon)\right\} \tag{21.27}$$

and put $t^* = t_1$, which is possible, since on the segment $[0, t_1]$, the point x (t) belongs to the domain D. But then, by virtue of eq.(21.24),

$$|x(t_1) - \xi(t_1)| < \frac{\rho}{2} + F(\varepsilon) = \rho - 2\delta < \rho - \delta,$$

which contradicts eq.(21.26).

Thus, we may put $t^* = \frac{L}{\varepsilon}$, so that the inequalities (21.24) are valid in the internal $0 < t < \frac{L}{\varepsilon}$, which completes the proof of our theorem.

We note now if the domain D is bounded (lies in a bounded part of the Euclidean space under consideration) then the requirement of uniformity may be excluded from the condition b), which now may be formulated as the condition that a limit (21.4) must exist at each point of this region.

In view of condition a), the functions

$$F_T(x) = \frac{1}{T}\int_0^T X(t,x)\,dt$$

satisfy the inequality

$$|F_T(x') - F_T(x'')| \leqslant \lambda |x' - x''|$$

and thus the sequence of these functions, as $T \to \infty$, is equicontinuous. However, since the domain D, being bounded, is compact, every equicontinuous sequence converging at each point of D will at the same time be uniformly convergent.

We note further that, since for every quasi-periodic function f (t) there exists a limit

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$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\,dt,$$

then condition b) is satisfied in the case where the domain D is bounded, if the expression X (t, x) for each x of D is a quasi-periodic function of the variable t.

We have here considered the question of the error of first approximation. However, it is not difficult to obtain an asymptotic evaluation of the error for the higher approximations.

Section 22. Neighborhoods of Points of Equilibrium and of Closed Orbits

Let us discuss now the problems of the second type.

Let us begin with the consideration of the simplest case, when the equations of first approximation (21.2) have a "quasi-static" solution corresponding to the "point of equilibrium"

$$\xi = \xi^0; \qquad X_0(\xi^0) = 0. \tag{22.1}$$

Then, for the solution of these equations, infinitely near to ξ^0, we have the equations of variation

$$\frac{d\delta\xi}{d\tau} = H\delta\xi, \qquad H = \left(\frac{\partial X_0(\xi)}{\partial\xi}\right)_0, \qquad \tau = \varepsilon t, \tag{22.2}$$

which are homogeneous linear differential equation with constant coefficients.

Consider the corresponding characteristic equation

$$\mathrm{Det}\,|pI - H| = 0 \tag{22.3}$$

and present the general solution for $\delta\xi$ (t) in the form

$$\delta\xi(t) = \sum_{(1\leq s\leq n)} C_s u_s(\tau),$$

where C_s are arbitrary constants, and u_s (τ) are linearly independent partial solutions corresponding to separate roots of the characteristic equation (22.3)

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For the simple root $p = p_s$,

$$u_s(\tau) = e^{p_s \tau}.$$

If this root is multiple, then

$$u_s(\tau) = P_s(\tau) e^{p_s \tau},$$

while $P_s(\tau)$ will be a polynomial in τ of a degree not higher than the multiplicity of p_s.

Thus, if all the roots of the given characteristic equation have negative real parts, then $\delta\xi$ exponentially tends toward zero.

Now let s of them have negative real parts, while all the remaining n − s roots have positive real parts.

Let us consider in this case the s-dimensional manifold $\mathfrak{M}_{t_0}$:

$$\delta\xi(t) = \sum_{(1 \leq j \leq s)} C_j u_j(\tau).$$

It is clear that if $\delta\xi(t_0)$ lies on $\mathfrak{M}_{t_0}$, then $\delta\xi(t)$ exponentially tends toward zero; but if $\delta\xi(t_0)$ does not lie on this manifold, then $\delta\xi(t)$, beginning at a sufficiently great t, will move away from $\mathfrak{M}_{t_0}$ without limit.

In particular, when the real parts of all the roots of the characteristic equation are positive, the manifold $\mathfrak{M}_{t_0}$ degenerates to the point $\delta\xi(t) = 0$, and any nontrivial solution for $\delta\xi(t)$ will, with the passage of time, leave any given neighborhood of this point.

If the real parts of some roots are zero, then the linearized eqs. (22.2) possess solutions of the form

$$\delta\xi(t) = e^{i\nu\tau} \qquad (22.4)$$

with real terms of ν.

However, in this case an arbitrarily small change in the form of the equation, for example the introduction of nonlinear terms, may radically modify the behavior of the solutions and cause damping or increment of the oscillations of the type of eq. (22.4)

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On the other hand, if the real parts of all roots of the characteristic equation differ from zero, the behavior of the equation will prove to be less sensitive to the introduction of small additions. In this case, we are able to prove theorems establishing that the solutions of the exact equations (21.1), lying in the neighborhood of ξ_0, possess properties which constitute a natural generalization of the properties of the solutions of $\delta\xi$.

Of course, for exact equations, a singular solution, close to ξ^* but generally depending on time, will play the role of a quasi-static solution.

Even the "refined first approximation"

$$\xi = \xi^0 + \varepsilon \widetilde{X}(t, \xi^0) = \xi^0 + \varepsilon \sum_{\nu \neq 0} \frac{X_\nu(\xi^0)}{i\nu} e^{i\nu t}$$

will depend on time, and oscillations with external frequencies present in the expression

$$X(t, x) = \sum_\nu X_\nu(x) e^{i\nu t}.$$

will appear in it.

In order to prove the theorem on the behavior of the exact solutions in the neighborhood of ξ^*, it is required, besides the above conditions, that only the most general conditions but none of the real parts of the roots of the characteristic equation (22.3) must vanish.

Thus we assume that:

a) the function X (t, x) and its partial derivatives of first order with respect to x_k are bounded and uniformly continuous with respect to x in the domain

$$-\infty < t < \infty, \qquad x \in D_\rho,$$

where D_ρ is a certain ρ-neighborhood of the point ξ_0;

b) at each point of D_ρ

$$\frac{1}{T}\int_{t}^{t+T} X(t, x)\,dt \to X_0(x)$$

is uniform with respect to t in the interval $(-\infty, \infty)$.

Putting

$$x = \xi^0 + b, \tag{22.5}$$

we may represent the basic equation in the form:

$$\frac{db}{dt} = \varepsilon Hb + \varepsilon B(t,\ b), \tag{22.6}$$

where

$$\left.\begin{aligned} B(t,\ b) &= Z(t,\ \xi^0 + b) + X_0(\xi^0 + b) - \\ &\quad - X_0(\xi^0) - \left(\frac{\partial X(\xi)}{\partial \xi}\right)_0 (\xi - \xi^0), \\ Z(t,\ x) &= X(t,\ x) - X_0(x). \end{aligned}\right\} \tag{22.7}$$

In this case, by virtue of a) and b), in the ρ-neighborhood of the point $b = 0$, the functions $B(t, b)$, $Z(t, \xi^0 + b)$ and their partial derivatives of first order in b_k are bounded and are uniformly continuous with respect to b in the domain

$$-\infty < t < \infty, \ |b| \leqslant \rho. \tag{22.8}$$

Moreover, at each point of the neighborhood under consideration, we have, uniformly with respect to t:

$$\frac{1}{T}\int_{t}^{t+T} Z(t,\ \xi^0 + b)\,dt \to 0,$$

$$\frac{1}{T}\int_{t}^{t+T} B(t,\ b)\,dt \to B(b) = X_0(\xi^0 + b) - X_0(\xi^0) - X'_{0\xi}(\xi^0)(\xi - \xi^0).$$

However, since $B(b)$, together with its partial derivatives of the first order, vanish at $b = 0$, we see that

$$|B(b') - B(b'')| \leqslant \eta_1(\sigma)|b' - b''|, \quad \eta_1(\sigma) \underset{\sigma \to 0}{\longrightarrow} 0$$

for

$$|b'| \leqslant \sigma, \quad |b''| \leqslant \sigma, \quad \sigma \leqslant b.$$

It is also clear that, if X (t, x) has partial derivatives to the n^{th} order inclusive, which are bounded and uniformly continuous with respect to b in the region (22.8), then B (t, b) and Z (t, ξ^* + b) will also possess these properties.

Before proceeding to investigate the basic equations in the form of eq.(22.6), let us further discuss, by analogy, the situation in the case when the equations of first approximation have a periodic solution, in whose δ-neighborhood the function X (t, x) has the properties a) and b).

Let us represent this periodic solution in the form

$$x = \xi(\omega\tau), \tag{22.9}$$

where ξ (φ) is periodic in φ with the period 2π.

Let us now set up, for the equations of first approximation, the corresponding equations of variation.

We obtain the homogeneous system of linear differential equations with periodic coefficients:

$$\frac{d\,\delta\xi}{d\tau} = X'_{0x}\{\xi(\omega\tau)\}\,\delta\xi. \tag{22.10}$$

Since, by definition of the function ξ (φ) (22.9), we have identically

$$\omega\xi'(\varphi) = X_0\{\xi(\varphi)\}, \tag{22.11}$$

then a differentiation of this relation with respect to φ shows that, at an arbitrary constant δu_0, the expression

$$\delta\xi = \xi'_\varphi(\omega\tau)\,\delta u_0$$

is a solution of the equation of variation (22.10).

Bearing in mind the Floquet-Lyapunov theorem on the properties of linear differential homogeneous equations with periodic coefficients, a transformation of the

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type

$$\delta\xi_k = \xi'_k(\omega\tau)\,\delta u_0 + \sum_{q=1}^{n-1} A_{kq}(\omega\tau)\,\delta u_q \quad (k = 1, 2, \ldots, n), \tag{22.12}$$

in which the $A_{kq}(\varphi)$ are periodic functions of φ with the period of 2π, possessing continuous first derivatives, will reduce eq.(22.10) to the system of differential equations with constant coefficients

$$\frac{d\delta u_0}{d\tau} = 0, \quad \frac{d\delta u_k}{d\tau} = \sum_{q=1}^{n-1} H_{kq}\,\delta u_q \qquad (k = 1, \ldots, n-1), \tag{22.13}$$

such that the roots of the equation:

$$\mathrm{Det}\,\|pI_{kq} - H_{kq}\| = 0; \quad I_{kq} = \begin{cases} 1, & k = q, \\ 0, & k \neq q, \end{cases} \tag{22.14}$$

become the characteristic exponents for eq.(22.10). In the transformation (22.10) the determinant

$$\begin{vmatrix} \xi'_1(\varphi) \ldots & \xi'_n(\varphi) \\ A_{11}(\varphi) \ldots & A_{n1}(\varphi) \\ \cdots & \cdots \\ A_{1,\,n-1}(\varphi) \ldots & A_{n,\,n-1}(\varphi) \end{vmatrix} \tag{22.15}$$

does not vanish for any value of φ; since this is a continuous periodic function of φ, it is possible to find a positive constant smaller than the modulus of this determinant for all values of φ.

It is convenient to return to the system of matrix-vectorial notation adopted by us. We introduce for this purpose the matrix $A(\varphi) = \| A_{kq}(\varphi) \|$ of n lines and $n-1$ columns, the square matrix of the $(n-1)^{th}$ order $H = \| H_{kq} \|$, and the vector δu with the components $\delta u_1, \ldots, \delta u_{n-1}$.

Then the transformation (22.12) and the equations (22.13) can be presented in the form

$$\delta\xi = \xi'(\omega\tau)\,\delta u_0 + A(\omega\tau)\,\delta u, \tag{22.16}$$

$$\frac{d\delta u_0}{d\tau} = 0; \quad \frac{d\delta u}{d\tau} = H\,\delta u. \tag{22.17}$$

On substituting eq.(22.16) in eq.(22.10), we obtain from eq.(22.17), the following identity:

$$\frac{\partial A(\varphi)}{\partial \varphi}\omega + A(\varphi)\,H = X'_{0x}\{\xi(\varphi)\}. \tag{22.18}$$

Having established this identity, let us return to eq.(21.1), writing it in the form

$$\frac{dx}{dt} = \varepsilon X_0(x) + \varepsilon Z(t,\, x), \tag{22.19}$$

where

$$Z(t,\, x) = X(t,\, x) - X_0(x). \tag{22.20}$$

Let us introduce here the new variables φ, b ($b_1, \ldots, b_{n-1}$) by means of the formulas

$$x = \xi(\varphi) + A(\varphi)\,b. \tag{22.21}$$

On substituting eq.(22.21) in eq.(22.19), we obtain

$$\begin{aligned}
&\xi'(\varphi)\frac{d\varphi}{dt} + \frac{d\varphi}{dt}A'(\varphi)\,b + A(\varphi)\frac{db}{dt} = \\
&\quad = \varepsilon X_0\{\xi + Ab\} + \varepsilon Z(t,\, \xi + Ab) = \varepsilon X_0\{\xi\} + \varepsilon X'_{0x}\{\xi\}\,Ab + \\
&\qquad + \varepsilon\,[X_0\{\xi + Ab\} - X_0\{\xi\} - X'_{0x}\{\xi\}\,Ab] + \varepsilon Z\{t,\, \xi + Ab\},
\end{aligned}$$

whence, by virtue of eqs.(22.11) and (22.18), we have

$$\begin{aligned}
&\{\xi'(\varphi) + A'(\varphi)\,b\}\left(\frac{d\varphi}{dt} - \varepsilon\omega\right) + A(\varphi)\left\{\frac{db}{dt} - \varepsilon Hb\right\} = \\
&\qquad = \varepsilon\,[X_0\{\xi + Ab\} - X_0\{\xi\} - X'_{0x}\{\xi\}\,Ab] + \\
&\qquad\qquad + \varepsilon Z\{t,\, \xi + Ab\}.
\end{aligned} \tag{22.22}$$

This relation, considered as a system of linear inhomogeneous equations in n unknowns,

$$\frac{d\varphi}{dt} - \varepsilon\omega; \quad \left\{\frac{db}{dt} - \varepsilon Hb\right\}, \ldots, \left\{\frac{db}{dt} - \varepsilon Hb\right\}_{n-1} \qquad (22.23)$$

has a determinant which coincides with the determinant (22.15) for $b = 0$.

Solving eq. (22.22) with respect to eq. (22.23), we find

$$\left.\begin{aligned} \frac{d\varphi}{dt} &= \varepsilon\omega + \varepsilon W(t, \varphi, b), \\ \frac{db}{dt} &= \varepsilon Hb + \varepsilon B(t, \varphi, b), \end{aligned}\right\} \qquad (22.24)$$

where

$$\left.\begin{aligned} W(t, \varphi, b) = K(\varphi, b)[X_0\{\xi(\varphi) + A(\varphi)b\} - \\ - X_0\{\xi(\varphi)\} - X'_{0x}\{\xi(\varphi)\}A(\varphi)b] + \\ + L(\varphi, b)Z\{t, \xi(\varphi) + A(\varphi)b\}, \\ B(t, \varphi, b) = M(\varphi, b)[X_0\{\xi(\varphi) + A(\varphi)b\} - \\ - X_0\{\xi(\varphi)\} - X'_{0x}\{\xi(\varphi)\}A(\varphi)b] + \\ + N(\varphi, b)Z\{t, \xi(\varphi) + A(\varphi)b\}. \end{aligned}\right\} \qquad (22.25)$$

Here K, L, M, and N are rational functions of b, regular for $b = 0$. Their coefficients in powers of b are continuous periodic functions of φ with a period of 2π and have continuous first derivatives with respect to φ. Let us put, for brevity,

$$\left.\begin{aligned} W(\varphi, b) = K(\varphi, b)[X_0\{\xi(\varphi) + A(\varphi)b\} - \\ - X_0\{\xi(\varphi)\} - X'_{0x}\{\xi(\varphi)\}A(\varphi)b], \\ B(\varphi, b) = M(\varphi, b)[X_0\{\xi(\varphi) + A(\varphi)b\} - \\ - X_0\{\xi(\varphi)\} - X'_{0x}\{\xi(\varphi)\}A(\varphi)b]. \end{aligned}\right\} \qquad (22.26)$$

We will denote the δ-neighborhood of the point $b = 0$ by U_δ. The domain of points (φ, b), for which b lies in U_δ will be denoted by ΩU_δ.

It is not hard to see that a positive number ε_0 can be found, so small that,

in the domain ΩU_δ,

$$|A(\varphi)b| < \rho$$

and that, in this domain, the functions K (φ, b); L (φ, b); M (φ, b); and N (φ, b) are bounded and continuous, together with all their first-order partial derivatives. Then, on the basis of eqs. (22.24) and (22.25) it follows that the functions

$$W(t, \varphi, b); \quad B(t, \varphi, b)$$

and their partial derivatives of the first order with respect to (φ, b) will be bounded and uniformly continuous with respect to (φ, b) in the domain

$$-\infty < t < \infty; \quad (\varphi, b) \in \Omega U_\delta. \tag{22.27}$$

Moreover, we have, at any point of the domain, uniformly with respect to t:

$$\frac{1}{T}\int_t^{t+T} W(t, \varphi, b)\,dt \to W(\varphi, b); \quad \frac{1}{T}\int_t^{T} B(t,\varphi,b)\,dt \to B(b, \varphi). \tag{22.28}$$

However, as indicated in eq. (22.26), the functions W (φ, b), B (φ, b) vanish

$$W(\varphi, 0) = 0, \quad B(\varphi, 0) = 0$$

together with their first-order partial derivatives, at b = 0.

Thus, taking an arbitrary positive $\sigma < \delta$, we arrive at the following inequalities, which are true in the domain ΩU_δ:

$$\left.\begin{aligned} |W(\varphi', b') - W(\varphi'', b'')| &\leqslant \eta_1(\sigma)\{|\varphi' - \varphi''| + |b' - b''|\}, \\ |B(\varphi', b') - B(\varphi'', b'')| &\leqslant \eta_1(\sigma)\{|\varphi' - \varphi''| + |b' - b''|\}, \end{aligned}\right\} \tag{22.29}$$

in which $\eta(\sigma) \to 0$ as $\sigma \to 0$.

It is clear, finally, that the functions W (t, φ, b), W (φ, b), B (t, φ, b), and B (φ, b) are periodic with the period 2π with respect to φ.

As shown above, eq. (22.6) is obtained from the system (22.24) in the case of degeneration, when B (t, φ, b) does not depend on φ.

For this reason, we will conduct our investigation with respect to the more general equations (22.24). Proceeding then to the case of degeneration, we will

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obtain results relating to eq. (22.6).

In the more general case, used here, we will assume that the real parts of all (n − 1) roots of the characteristic equation (22.14) are different from zero.

Thereby we impose the requirement that the real parts of the (n − 1) characteristic exponents* for the equations of variation (22.10) do not vanish.

For convenience in studying eqs. (22.24) it is expedient to transform them to the following form:

$$\frac{dg}{dt} = \omega(\varepsilon) + P(t, g, h, \varepsilon),$$

$$\frac{dh}{dt} = Hh + Q(t, g, h, \varepsilon),$$

so that P and Q are sufficiently small for small h and ε.

This investigation will be described below.

Consider a certain function f (t, x) defined for all real t and for all x of the set E. Assume that E is a complex set of a certain metric space, and that, at each point E is uniform with respect to t:

$$\frac{1}{T}\int_{t}^{t+T} f(t, x)\,dt \to 0; \quad T \to \infty. \tag{22.30}$$

We shall also assume that positive constants M, λ may be assigned, such that, for all real values of t, and for all values of x, x', x'' of E, the inequalities

$$|f(t, x)| \leqslant M; \quad |f(t, x') - f(t, x'')| \leqslant \lambda\rho(x', x''), \tag{22.31}$$

are valid, where ρ (x', x'') denotes the distance between the points x' and x''. Obviously, it follows from the conditions adopted that the relation (22.30) holds uniformly not only with respect to t, but also with respect to (t, x). We may therefore construct a function ε (T), tending toward zero as T → ∞, such that

* The n^{th} characteristic exponent in this case always vanishes.

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$$\left|\frac{1}{T}\int_{t}^{t+T} f(t,\ x)\,dt\right| \leqslant \varepsilon(T); \quad -\infty < t < \infty; \quad x \in E. \qquad (22.32)$$

Let us now take an arbitrary η and construct the function

$$f_{\eta}(t,\ x) = \int_{-\infty}^{t} e^{-\eta(t-\tau)} f(\tau,\ x)\,d\tau. \qquad (22.33)$$

We then have

$$f_{\eta}(t,\ x) = \int_{0}^{\infty} e^{-\eta z} f(t-z,\ x)\,dz =$$
$$= \sum_{n=0}^{\infty} e^{-\eta nT} \int_{nT}^{(n+1)T} f(t-z,\ x)\, e^{-\eta(z-nT)}\,dz,$$

Therefore, on the basis of eq.(22.31), we obtain

$$|f_{\eta}(t,\ x)| \leqslant \sum_{n=0}^{\infty} e^{-\eta nT} \left|\int_{nT}^{(n+1)T} f(t-z,\ x)\,dz\right| +$$
$$+ M \sum_{n=0}^{\infty} e^{-\eta nT} \int_{nT}^{(n+1)T} (1 - e^{-\eta(z-nT)})\,dz \leqslant$$
$$\leqslant \sum_{n=0}^{\infty} e^{-\eta nT} \left|\int_{nT-t}^{nT-t+T} f(t,\ x)\,dt\right| + MT,$$

or, in view of eq.(22.32),

$$|f_{\eta}(t,\ x)| < \sum_{n=0}^{\infty} e^{-\eta nT} \varepsilon(T)\, T + MT = \frac{T\varepsilon(T)}{1 - e^{-\eta T}} + MT. \qquad (22.34)$$

Up to now the quantity T has been arbitrary. Let us now take T as a function of η, determined by the equation

$$1 - e^{-\eta T} = \varepsilon(T).$$

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Since $\varepsilon(T) \to 0$ as $T \to \infty$, it is obvious that, for T_η, determined by this equation, the relation $\eta T_\eta \to 0$ as $\eta \to 0$ is obtained.

Assuming that

$$(M+1)\eta T_\eta = \zeta(\eta),$$

eq. (22.34) indicates that, for the function f (t, x) under consideration, the following inequality holds:

$$\left|\int_{-\infty}^{t} e^{-\eta(t-\tau)} f(\tau, x)\, d\tau\right| \leqslant \frac{\zeta(\eta)}{\eta}; \quad -\infty < t < \infty; \ x \in E, \tag{22.35}$$

in which

$$\zeta(\eta) \to 0 \quad \text{as} \quad \eta \to 0. \tag{22.36}$$

Let us now apply the results to the case of the functions

$$\left.\begin{aligned} W^*(t, \varphi, b) &= W(t, \varphi, b) - W(\varphi, b), \\ B^*(t, \varphi, b) &= B(t, \varphi, b) - B(\varphi, b), \end{aligned}\right\} \tag{22.37}$$

taking the domain ΩU_δ as the set E.

Since these functions are periodic in φ with a period of 2π, then obviously we can represent Ω as a circle, so that $E = \Omega U_\delta$ will be a compact domain in the metrical space constituting the topological product of Ω and (n − 1)-dimensional Euclidean space. We may therefore construct a function $\zeta(\eta)$ having the property (22.36), in such a way that, for an arbitrary positive η,

$$\left.\begin{aligned} |W^*_\eta(t, \varphi, b)| \leqslant \frac{\zeta(\eta)}{\eta}; \quad |B^*_\eta(t, \varphi, b)| \leqslant \frac{\zeta(\eta)}{\eta}; \\ -\infty < t < \infty; \quad (\varphi, b) \in \Omega U_\delta, \end{aligned}\right\} \tag{22.38}$$

where

$$\left.\begin{aligned} W^*_\eta(t, \varphi, b) &= \int_{-\infty}^{t} e^{-\eta(t-\tau)} W^*(\tau, \varphi, b)\, d\tau, \\ B^*_\eta(t, \varphi, b) &= \int_{-\infty}^{t} e^{-\eta(t-\tau)} B^*(\tau, \varphi, b)\, d\tau. \end{aligned}\right\} \tag{22.39}$$

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We now introduce, by analogy with the definition (21.5) in the proof of theorem I, the function

$$\Delta_a(b) = \begin{cases} A_a\left\{1 - \frac{|b|^2}{a^2}\right\}^{2q}, & |b| \leqslant a, \\ 0, & |b| > a, \end{cases} \tag{22.40}$$

where a is a certain sufficiently small number, while A_a is determined by the condition of norming

$$\int\limits_{U_\delta} \Delta_a(b)\, db = 1, \quad db = db_1 \ldots db_{n-1}. \tag{22.41}$$

Here $q \geq 1$ is a certain fixed integer whose value may be taken as large as desired.

Let us also construct the function $\delta_a(\varphi)$ of a single real variable, assigning it over the interval $(-\pi, \pi)$ by the aid of the relations:

$$\left.\begin{aligned} &\delta_a(\varphi) = \Phi_a\left\{1 - \frac{\varphi^2}{a^2}\right\}^{2q}, \quad |\varphi| \leqslant a, \\ &\delta_a(\varphi) = 0, \quad |\varphi| > a, \\ &\Phi_a \int\limits_{-a}^{a} \left\{1 - \frac{\varphi^2}{a^2}\right\}^{2q} d\varphi = 1, \quad a < \pi \end{aligned}\right\} \tag{22.42}$$

and extending its domain of definition to the whole real axis by means of the condition of periodicity with a period of 2π.

On introducing these functions, we construct the expressions

$$\left.\begin{aligned} u(t, \varphi, b) &= \int\limits_{\Omega}\int\limits_{U_\delta} \delta_a(\varphi - \varphi')\,\Delta_a(b - b')\,W_\varepsilon^*(t, \varphi', b')\,d\varphi'\,db', \\ v(t, \varphi, b) &= \int\limits_{\Omega}\int\limits_{U_\delta} \delta_a(\varphi - \varphi')\,\Delta_a(b - b')\,B_\varepsilon^*(t, \varphi', b')\,d\varphi'\,db', \end{aligned}\right\} \tag{22.43}$$

which obviously possess the period 2π with respect to φ.

We note that its construction will cause the function

$$\delta_a(\varphi - \varphi')\Delta_a(b - b') \qquad (22.44)$$

to possess partial derivatives with respect to φ, b, up to the order 2q inclusive, while the function (22.44) itself, like all its partial derivatives up to the order 2q inclusive, are bounded and, according to the norm, will not exceed a certain quantity $G(a)$, which generally tends toward ∞ as $a \to 0$. Hence, on the basis of eq.(22.38) we may conclude that the function (22.43) and all of its partial derivatives with respect to φ, b to the $2q^{th}$ order inclusive are bounded according to the norm on the set* $R\Omega U_\delta$ by the quantity $G(a)\frac{\zeta(\eta)}{\eta}$.

Up to now, a and η have been arbitrary. Let us now introduce, as a and η, certain functions a_ε, η_ε of the parameter ε such that

$$a_\varepsilon \to 0; \quad \eta_\varepsilon \to 0; \quad \varepsilon G(a_\varepsilon)\frac{\zeta(\eta_\varepsilon)}{\eta_\varepsilon} \to 0;$$
$$G(a_\varepsilon)\zeta(\eta_\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0. \qquad (22.45)$$

Let us fix a certain positive $\rho_0 < \delta$ and let us take ε^* so small that for $0 < \varepsilon < \varepsilon^*$, the value $\rho_0 + a < \delta$ will hold. Then, by the definition of eqs.(22.40), (22.41) of the function $\Delta(b)$, we see that

$$\int_{U_\delta} \Delta_a(b - b')\,db' = 1; \quad b \in U_{\rho_0}, \quad 0 < \varepsilon < \varepsilon^*. \qquad (22.46)$$

We note that eqs.(22.39) and (22.46) yield, identically,

$$\frac{\partial u}{\partial t} + \eta u = \int_\Omega \int_{U_\delta} \delta_a(\varphi - \varphi')\Delta_a(b - b')W^*(t, \varphi', b')\,d\varphi'\,db',$$

whence

$$\frac{\partial u}{\partial t} + \eta u - W^*(t, \varphi, b) =$$

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* For brevity we here denote the real axis $(-\infty, \infty)$ by R, so that $R\Omega U_\delta$ denotes the set of points (t, φ, b) for which $-\infty < t < \infty$, $\varphi \in \Omega$, $b \in U_\delta$.

$$= \int\limits_{\Omega}\int\limits_{\Gamma_\delta} \delta_a(\varphi - \varphi')\Delta_a(b - b')\{W^*(t, \varphi', b') - - W^*(t, \varphi, b)\}\, d\varphi' db' \tag{22.47}$$

for

$$(t, \varphi, b) \in R\Omega U_{\rho_0}, \qquad 0 < \varepsilon < \varepsilon^*,$$

since, in view of eq. (22.46) and the periodicity of the function $\delta_a(\varphi)$,

$$\int\limits_{\Omega}\int\limits_{\Gamma_\delta} \delta_a(\varphi - \varphi')\Delta_a(b - b')\, d\varphi' db' = \int\limits_{\Omega} \delta_a(\varphi - \varphi')\, d\varphi' = \int\limits_{\Omega} \delta_a(\varphi')\, d\varphi' = 1.$$

On the other hand, we note that a positive λ can be assigned, such that the inequality

$$|W^*(t, \varphi', b') - W^*(t, \varphi, b)| \leqslant \lambda\{|\varphi' - \varphi| + |b' - b|\},$$
$$(t, \varphi, b),\ (t, \varphi', b') \in R\Omega U_\delta.$$

shall be satisfied.

Further, owing to the periodicity of $\delta_a(\varphi)$ with respect to φ, an integration with respect to φ' can be performed over any interval of length 2π; therefore, in particular, the interval φ / π, $\varphi + \pi$ may be taken as the interval of integration. With such a selection, $\varphi - \varphi'$ will vary from $-\pi$ to π and, consequently, $\delta_a(\varphi - \varphi')$ will fail to vanish only when

$$|\varphi - \varphi'| \leqslant a.$$

We therefore have, on $R\Omega U_{\rho o}$, for $0 < \varepsilon \omega \varepsilon$:

$$\left|\frac{\partial u}{\partial t} + \gamma_i u - W^*(t, \varphi, b)\right| \leqslant$$
$$\leqslant 2\lambda a \int\limits_{\Omega}\int\limits_{\Gamma_\delta} \delta_a(\varphi - \varphi')\Delta_a(b - b')\, d\varphi' db' = 2\lambda a. \tag{22.48}$$

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Now consider the partial derivatives:

$$\frac{\partial}{\partial b_k}\left\{\frac{\partial u}{\partial t}+\tau_1 u - W^*(t,\ \varphi,\ b)\right\};$$

$$\frac{\partial}{\partial \varphi}\left\{\frac{\partial u}{\partial t}+\tau_1 u - W^*(t,\ \varphi,\ b)\right\} \qquad (22.49)$$

and note that, by integrating eq. (22.47) in steps, they may be represented, respectively, in the form

$$\int_{\Phi}\int_{U_\delta}\delta_\alpha(\varphi-\varphi')\Delta_\alpha(b-b')\left\{\frac{\partial W^*(t,\ \varphi',\ b')}{\partial b'_k}-\frac{\partial W^*(t,\ \varphi,\ b)}{\partial b_k}\right\}d\varphi'\,db',$$

$$\int_{\Phi}\int_{U_\delta}\delta_\alpha(\varphi-\varphi')\Delta_\alpha(b-b')\left\{\frac{\partial W^*(t,\ \varphi',\ b)}{\partial \varphi'}-\frac{\partial W^*(t,\ \varphi,\ b)}{\partial \varphi}\right\}d\varphi'\,db'.$$

However, in view of the continuity of the partial derivatives entering into them, a monotonously decreasing function $\xi(\varepsilon)$, tending toward zero as $\varepsilon \to 0$ can be found, such that

$$\left|\frac{\partial W^*(t,\varphi',b')}{\partial b'_k}-\frac{\partial W^*(t,\ \varphi,\ b)}{\partial b_k}\right| \leqslant \xi\{|\varphi'-\varphi|+|b'-b|\},$$

$$\left|\frac{\partial W^*(t,\varphi',b')}{\partial \varphi'}-\frac{\partial W^*(t,\ \varphi,\ b)}{\partial \varphi}\right| \leqslant \xi\{|\varphi'-\varphi|+|b'-b|\},$$

so that, reasoning as above, it can be demonstrated that the derivatives (22.49) on the set $R\Omega U_{\rho 0}$ will be, according to the norm, less than $2\xi a$ for $0 < \varepsilon < \varepsilon$.

Since, as $\varepsilon \to 0$, we have $a \to 0$, it is obvious that the function (22.47) and its partial derivatives with respect to a, φ on the set $R\Omega U_{\rho 0}$ will, according to the norm, be less than a certain quantity $\mu(\varepsilon)$ which tends toward zero together with ε.

In a completely analogous manner we can prove that the same property is also exhibited by the function

$$\frac{\partial v}{\partial t}+\tau_1 v - B^*(t,\ \varphi,\ b).$$

On the other hand, the functions ηu, ηv and their partial derivatives with respect to φ, b are bounded, according to the norm, on the set $R\Omega U_{\rho_0}$, by the quantity $G(\varepsilon)\eta$ of eq. (22.45), which tends toward zero as $\varepsilon \to 0$.

Thus the functions

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - W^*(t, \varphi, b) = \frac{\partial u}{\partial t} - W(t, \varphi, b) + W(\varphi, b), \\ \frac{\partial v}{\partial t} - B^*(t, \varphi, b) = \frac{\partial v}{\partial t} - B(t, \varphi, b) + B(\varphi, b) \end{aligned}\right\} \quad (22.50)$$

and their partial derivatives of the first order with respect to φ, b are bounded, according to the norm, on the set $R\Omega U_{\rho_0}$ by the quantity $\alpha(\varepsilon)$, tending toward zero together with ε.

Noting this, let us return to eq. (22.24) and perform the substitution of variables, by using

$$\varphi = g + \varepsilon u(t, g, h); \quad b = h + \varepsilon v(t, g, h). \quad (22.51)$$

Differentiating eq. (22.51) and substituting in eq. (22.24), we get

$$\frac{dg}{dt} + \varepsilon\frac{\partial u}{\partial g}\frac{dg}{dt} + \varepsilon\frac{\partial u}{\partial h}\frac{dh}{dt} =$$

$$= -\varepsilon\frac{\partial u}{\partial t} + \varepsilon\omega + \varepsilon W\{t, g + \varepsilon u, h + \varepsilon v\},$$

$$\frac{dh}{dt} + \varepsilon\frac{\partial v}{\partial g}\frac{dg}{dt} + \varepsilon\frac{\partial v}{\partial h}\frac{dh}{dt} =$$

$$= -\varepsilon\frac{\partial v}{\partial t} + \varepsilon Hh + \varepsilon^2 Hv + \varepsilon B\{t, g + \varepsilon u, h + \varepsilon v\},$$

or

$$\begin{aligned} &\frac{dg}{dt} + \varepsilon\frac{\partial u}{\partial g}\frac{dg}{dt} + \varepsilon\frac{\partial u}{\partial h}\frac{dh}{dt} = \\ &= -\varepsilon\left\{\frac{\partial u}{\partial t} - W(t, g, h) + W(g, h)\right\} + \\ &+ \varepsilon\{W(t, g + \varepsilon u, h + \varepsilon v) - W(t, g, h)\} + \\ &\qquad + \varepsilon W(g, h) + \varepsilon\omega, \end{aligned}$$

$$\left.\begin{aligned}\frac{dh}{dt} + \varepsilon\frac{\partial v}{\partial g}\frac{dg}{dt} + \varepsilon\frac{\partial v}{\partial h}\frac{dh}{dt} &= \\ = -\varepsilon\left\{\frac{\partial v}{\partial t} - B(t,\ g,\ h) + B(g,\ h)\right\} &+ \\ + \varepsilon^2 Hv + \varepsilon\{B(t, g+\varepsilon u,\ h+\varepsilon v) & \\ - B(t,\ g,\ h)\} + \varepsilon B(g,\ h) &+ \varepsilon Hh.\end{aligned}\right| \quad (22.52)$$

Let us now take $\varepsilon_1 < \varepsilon^*$ so small that, for every positive value of ε not exceeding ε_1, the inequality

$$\varepsilon\frac{\zeta(\eta)}{\eta} < \delta - \rho_1, \text{ where } 0 < \rho_1 < \delta.$$

will be valid.

Then, since eq. (22.38) shows that

$$|\varepsilon v| \leqslant \varepsilon\frac{\zeta(\eta)}{\eta},$$

then for $0 < \varepsilon \leq \varepsilon_1$, $(t,\ g,\ h)$ $R\Omega U_{\rho 1}$ we will have

$$h + \varepsilon v \in U_\delta.$$

From this we prove that the functions

$$\begin{gathered}W(t,\ g+\varepsilon u,\ h+\varepsilon v) - W(t,\ g,\ h);\\ B(t,\ g+\varepsilon u,\ h+\varepsilon v) \quad B(t,\ g,\ h)\end{gathered}$$

and their partial derivatives of the first order with respect to g, h are bounded according to the norm, on the set $R\Omega U_{\rho 1}$, by a certain function tending toward zero with ε. For this reason, in view of the previously mentioned properties of the function (22.50), the expressions

$$\left.\begin{aligned}L_1(t,\ g,\ h,\ \varepsilon) &= W(t,\ g+\varepsilon u,\ h+\varepsilon v) - W(t,\ g,\ h) - \\ &- \frac{\partial u}{\partial t} + W(t,\ g,\ h) - W(g,\ h),\\ L_2(t,\ g,\ h,\ \varepsilon) &= B(t,\ g+\varepsilon u,\ h+\varepsilon v) - B(t,\ g,\ h) - \\ &- \frac{\partial v}{\partial t} + B(t,\ g,\ h) - B(g,\ h) + \varepsilon Hv,\end{aligned}\right\} \quad (22.53)$$

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and their partial derivatives of the first order with respect to g, h are bounded according to the norm, on the set $R\Omega U_{\rho 1}$ by a certain function of ε, tending toward zero as $\varepsilon \to 0$. Noting this, let us represent eq. (22.52) in a form solvable with respect to the derivatives $\frac{dg}{dt}$, $\frac{dh}{dt}$, for which purpose let us consider the inverse of the matrix

$$\begin{Vmatrix} 1_1 + \varepsilon \dfrac{\partial u}{\partial g} & \varepsilon \dfrac{\partial u}{\partial h} \\ \varepsilon \dfrac{\partial v}{\partial g} & 1_{n-1} + \varepsilon \dfrac{\partial v}{\partial h} \end{Vmatrix}, \tag{22.54}$$

where 1_{n-1} represents the square unit matrix of the $(n-1)^{th}$ order.

Since, according to the above statements,

$$\varepsilon \frac{\partial u}{\partial g}, \quad \varepsilon \frac{\partial u}{\partial h}, \quad \varepsilon \frac{\partial v}{\partial g}, \quad \varepsilon \frac{\partial v}{\partial h}$$

and their partial derivatives of the first order with respect to g, h are bounded, according to the norm, on the set $R\Omega U_{\rho 1}$ by the functions $\alpha(\varepsilon)$, tending toward zero as $\varepsilon \to 0$, it follows that a positive $\varepsilon_0 < \varepsilon_1$ can be assigned for which any positive $\varepsilon < \varepsilon_0$ of the inverse of matrix (22.54) exists everywhere on $R\Omega U_{\rho 1}$, and may be represented in the form

$$\begin{Vmatrix} 1_1 + \mathfrak{A}_{11} & \mathfrak{A}_{12} \\ \mathfrak{A}_{21} & 1_{n-1} + \mathfrak{A}_{22} \end{Vmatrix},$$

where

$$\mathfrak{A}_{11}(t, g, h, \varepsilon), \quad \mathfrak{A}_{12}(t, g, h, \varepsilon), \quad \mathfrak{A}_{21}(t, g, h, \varepsilon), \quad \mathfrak{A}_{22}(t, g, h, \varepsilon) \tag{2.55}$$

and their partial derivatives of the first order with respect to g, h tend toward zero as $\varepsilon \to 0$ uniformly on $R\Omega U_{\rho 1}$.

From eq. (22.52) we have

$$\frac{dg}{dt} = \varepsilon\omega + \varepsilon \Pi(t, g, h, \varepsilon); \qquad \frac{dh}{dt} = \varepsilon Hh + \varepsilon \Gamma(t, g, h, \varepsilon). \tag{22.56}$$

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where

$$\left.\begin{aligned}&\Pi(t, g, h, \varepsilon) = W(g, h) + \mathfrak{A}_{11}(t, g, h, \varepsilon)\times\\&\times\{\omega + W(g, h) + L_1(t, g, h, \varepsilon)\} + \mathfrak{A}_{12}(t, g, h, \varepsilon)\times\\&\quad\times\{Hh + B(g, h) + L_2(t, g, h, \varepsilon)\} + L_1(t, g, h, \varepsilon);\\&\Gamma(t, g, h, \varepsilon) = B(g, h) + \mathfrak{A}_{21}(t, g, h, \varepsilon)\times\\&\times\{\omega + W(g, h) + L_1(t, g, h, \varepsilon)\} + \mathfrak{A}_{22}(t, g, h, \varepsilon)\times\\&\quad\times\{Hh + B(g, h) + L_2(t, g, h, \varepsilon)\} + L_2(t, g, h, \varepsilon).\end{aligned}\right\} \tag{22.57}$$

Here we will make several remarks on the properties of the functions Π, Γ introduced by us, which will be necessary for the further discussion.

As indicated by eq.(22.57), the functions Π, Γ are defined for each positive value of $\varepsilon < \varepsilon_0$ on the set $R\Omega U_{\rho_1}$, and on this set the functions

$$\Pi(t, g, h, \varepsilon) - W(g, h); \quad \Gamma(t, g, h, \varepsilon) - B(g, h)$$

and their partial derivatives with respect to g, h tend toward zero uniformly as $\varepsilon \to 0$.

Taking account of eq.(22.29), we conclude that we may assign functions of $M(\varepsilon)$; $\lambda(\varepsilon, \sigma)$

$$M(\varepsilon) \to 0 \text{ where } \varepsilon \to 0; \quad \lambda(\varepsilon, \sigma) \to 0 \text{ where } \varepsilon \to 0, \ \sigma \to 0,$$

such that the following inequalities will hold:

$$|\Pi(t, g, 0, \varepsilon)| \leqslant M(\varepsilon); \quad |\Gamma(t, g, 0, \varepsilon)| \leqslant M(\varepsilon);$$
$$\varepsilon \leqslant \varepsilon_0, \quad t \in R, \quad g \in \Omega.$$
$$|\Pi(t, g', h', \varepsilon) - \Pi(t, g'', h'', \varepsilon)| \leqslant$$
$$\leqslant \lambda(\varepsilon, \sigma)\{|g' - g''| + |h' - h''|\};$$
$$(t, g', h'), (t, g'', h'') \in R\Omega U_\sigma, \quad 0 < \varepsilon < \varepsilon_0.$$
$$|\Gamma(t, g', h', \varepsilon) - \Gamma(t, g'', h'', \varepsilon)| \leqslant$$
$$\leqslant \lambda(\varepsilon, \sigma)\{|g' - g''| + |h' - h''|\},$$

where σ is an arbitrary positive number less than ρ_1.

Further, it is clear that Π, Γ have a period of 2π with respect to g. It is also clear that, if $\{\tau_m\}$ is a sequence of R for which, uniformly on $R\Omega U_\delta$:

$$\left.\begin{aligned} &|W(t+\tau_m, \varphi, b) - W(t, \varphi, b)| \to 0, \\ &|B(t+\tau_m, \varphi, b) - B(t, \varphi, b)| \to 0, \end{aligned}\right\} m \to \infty$$

then for this sequence, uniformly on $R\Omega U_{\rho 1}$, we will have

$$\begin{aligned} &|\Pi(t+\tau_m, g, h, \varepsilon) - \Pi(t, g, h, \varepsilon)| \to 0, \\ &|\Gamma(t+\tau_m, g, h, \varepsilon) - \Gamma(t, g, h, \varepsilon)| \to 0 \end{aligned}$$

for every ε such that $0 < \varepsilon \mu \varepsilon_0$.

Taking into account eqs.(22.37), (22.39), (22.43) and (22.51), it can be readily demonstrated that, when the condition a) is supplemented by the additional requirement that the partial derivatives to the m^{th} order inclusive shall be bounded and uniformly continuous, the above constructed functions Π, Γ will likewise possess bounded and uniformly continuous partial derivatives to the m^{th} order inclusive.

Returning to eq.(22.56), let us make in them the transition to the "slow time", by putting

$$\tau = \varepsilon t.$$

This will yield

$$\left.\begin{aligned} \frac{dg}{d\tau} &= \omega + \Pi\left(\frac{\tau}{\varepsilon}, g, h, \varepsilon\right), \\ \frac{dh}{d\tau} &= Hh + \Gamma\left(\frac{\tau}{\varepsilon}, g, h, \varepsilon\right). \end{aligned}\right\} \qquad (22.58)$$

We now distinguish one special case when the functions $W(t, \varphi, b)$, $B(t, \varphi, b)$, entering into eq.(22.24) have the form

$$\left.\begin{aligned} W(t, \varphi, b) &= \overline{W}(t, \varphi + \nu t, b), \\ B(t, \varphi, b) &= \overline{B}(t, \varphi + \nu t, b), \end{aligned}\right\} \qquad (22.59)$$

where $\overline{W}(t, \varphi, b)$, $\overline{B}(t, \varphi, b)$ are periodic in t with a certain constant period T.

Then, eqs.(22.37), (22.39), (22.43), and (22.51) indicate that

$$\Pi(t,\ g,\ h,\ \varepsilon) = \overline{\Pi}(t,\ g + \nu t,\ h,\ \varepsilon),$$
$$\Gamma(t,\ g,\ h,\ \varepsilon) = \overline{\Gamma}(t,\ g + \nu t,\ h,\ \varepsilon),$$

where $\overline{\Pi}$ and $\overline{\Gamma}$ will have the same period T with respect to t.

In the special case of eq.(22.59) under consideration, a transition to the new angular variable

$$g + \nu t = g + \frac{\nu\tau}{\varepsilon} = \vartheta$$

will furnish, instead of eq.(22.58), the equations

$$\left.\begin{aligned} \frac{d\vartheta}{d\tau} &= \omega + \frac{\nu}{\varepsilon} + \Pi\left(\frac{\tau}{\varepsilon},\ \vartheta,\ h,\ \varepsilon\right), \\ \frac{dh}{d\tau} &= Hh + \Gamma\left(\frac{\tau}{\varepsilon},\ \vartheta,\ h,\ \varepsilon\right), \end{aligned}\right\} \qquad (22.60)$$

whose right sides possess the period εT with respect to the independent variable τ.

Summarizing all above statements, we note that, under the assumptions made, our basic equation has been reduced to the form

$$\left.\begin{aligned} \frac{dg}{dt} &= G(\varepsilon) + P(t,\ g,\ h,\ \varepsilon), \\ \frac{dh}{dt} &= Hh + Q(t,\ g,\ h,\ \varepsilon). \end{aligned}\right\} \qquad (22.61)$$

In this case, we may assign positive numbers ε^*, ρ such that the following conditions will be satisfied:

a) the function G (ε) is defined for $0 < \varepsilon < \varepsilon^*$.

b) the functions P (t, g, h, ε); Q (t, g, h, ε) are defined in the domain

$$t \in R;\quad g \in \Omega;\quad h \in U_\rho;\quad 0 < \varepsilon < \varepsilon^*$$

and have the period 2π with respect to the angular variable g.

c) for $t \in R$, g Ω, $0 < \varepsilon < \varepsilon^*$, the following inequation will hold:

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$$|P(t, g, 0, \varepsilon)| \leqslant M(\varepsilon); \quad |Q(t, g, 0, \varepsilon)| \leqslant M(\varepsilon),$$

where $M(\varepsilon) \to 0$ as $\varepsilon \to 0$;

d) for any positive $\sigma < \rho$ in the domain

$$t \in R; \quad g' \in \Omega; \quad g'' \in \Omega; \quad h' \in U_\sigma; \quad h'' \in U_\sigma; \quad 0 < \varepsilon < \varepsilon^*$$

the following inequations will hold

$$\begin{aligned} |P(t, g', h', \varepsilon) - P(t, g'', h'', \varepsilon)| &\leqslant \\ &\leqslant \lambda(\varepsilon, \sigma)\{|g' - g''| + |h' - h''|\}, \\ |Q(t, g', h', \varepsilon) - Q(t, g'', h'', \varepsilon)| &\leqslant \\ &\leqslant \lambda(\varepsilon, \sigma)\{|g' - g''| + |h' - h''|\}, \end{aligned}$$

in which $\lambda(\varepsilon, \sigma) \to 0$ as $\varepsilon \to 0$, $\sigma \to 0$;

e) all the real parts of the roots $z_1, \ldots, z_{n-1}$ of the equation

$$\mathrm{Det}\,|zI_{n-1} - H| = 0$$

differ from zero.

We will now formulate and prove a series of assertions on the properties of the system of equations (22.61).

Argument I. A positive ε_0 may be assigned such that, for each positive ε less than ε_0, the system of equations (22.61) has an integral manifold representable by a relation of the form

$$h = f(t, g, \varepsilon), \tag{22.62}$$

in which $f(t, g, \varepsilon)$, as a function of (t, g), is defined on $R\Omega$ and satisfies the inequalities:

$$\left.\begin{aligned} &|f(t, g, \varepsilon)| \leqslant D(\varepsilon) < \rho; \\ &|f(t, g', \varepsilon) - f(t, g'', \varepsilon)| \leqslant \Delta(\varepsilon)|g' - g''|, \end{aligned}\right\} \tag{22.63}$$

and

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$$D(\varepsilon) \to 0, \quad \Delta(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

The function $f(t, g, \varepsilon)$ possesses the period 2π with respect to the angular variable g.

If the functions $P(t, g, h, \varepsilon)$, $Q(t, g, h, \varepsilon)$ in the domain

$$t \in R, \; g \in \Omega, \quad h \in U_p, \quad 0 < \varepsilon < \varepsilon^*$$

have bounded and uniformly continuous partial derivatives with respect to g, h to the m^{th} order inclusive, then $f(t, g, \varepsilon)$ will have bounded and uniformly continuous derivatives with respect to g to the m^{th} order inclusive.

Proof of Argument I. Consider the matrix H and represent it in the form:

$$H = U \begin{vmatrix} H_+ & 0 \\ 0 & H_- \end{vmatrix} U^{-1}, \tag{22.64}$$

where U is a matrix having the inverse U^{-1}; 0 is the null matrix; H_+, H_- are matrices for which the roots of the characteristic equations are roots of the equations reduced to condition e), with positive real parts for H_+ and negative real parts for H_-. Then it is obvious that the matrix $J(t)$, defined by the relations

$$\left.\begin{aligned} J(t) &= -U \begin{vmatrix} e^{-H_+ t} & 0 \\ 0 & 0 \end{vmatrix} U^{-1} \quad \text{for} \quad t > 0, \\ J(t) &= U \begin{vmatrix} 0 & 0 \\ 0 & e^{-H_- t} \end{vmatrix} U^{-1} \quad \text{for} \quad t < 0 \end{aligned}\right\} \tag{22.65}$$

satisfies the differential equations

$$\frac{dJ}{dt} = -HJ = -JH \quad \text{for} \quad t \neq 0 \tag{22.66}$$

and the condition of discontinuity at $t = 0$:

$$J(-0) - J(+0) = E. \tag{22.67}$$

Since the elements of the matrix $e^{-H_+ t}$ are, in the general case, sums of the products of polynomials in t by complex exponentials $e^{(\mu + i\nu) t}$ with negative values

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of μ, while the elements of the matrix $e^{-H_- t}$ are analogous combinations of polynomials and exponentials with positive values of μ, it is obvious that we can always find positive constants α, K for which the inequation

$$|J(t)| \leqslant Ke^{-\alpha|t|} \qquad (22.68)$$

will hold over the entire real axis.

Noting this, let us fix the positive numbers D, Δ ($\Delta < \rho$), and let us consider the class C (D, Δ) of functions F (t, g) with values from E_{n-1} defined on $R\Omega$, satisfying the inequations

$$|F(t, g)| \leqslant D; \ |F(t, g') - F(t, g'')| \leqslant \Delta |g' - g''| \qquad (22.69)$$

and having the period 2π with respect to the angular variable. Let us consider, for a certain function F (t, g) of the class C (D, Δ), the equation in the form

$$\frac{dg}{dt} = G(\varepsilon) + P(t, g, F(t, g), \varepsilon). \qquad (22.70)$$

By virtue of conditions c), d), everywhere on $R\Omega$ we have

$$\left.\begin{aligned} &|P(t, g', F(t, g')\varepsilon)| \leqslant M(\varepsilon) + \lambda(\varepsilon, D)D, \\ &|P(t, g', F(t, g'), \varepsilon) - P(t, g'', F(t, g''), \varepsilon)| \leqslant \\ &\qquad \leqslant \lambda(\varepsilon, D)(1 + \Delta)|g' - g''|. \end{aligned}\right\} \qquad (22.71)$$

For this reason, on assigning the arbitrary initial conditions

$$g = g^0 \quad \text{as} \quad t = t_0,$$

we can construct the solution of eq. (22.70) for any t. Let us denote it symbolically in the form

$$g_t = T^F_{z, t_0}(g^0), \quad \text{where} \quad z = t - t_0. \qquad (22.72)$$

It goes without saying that, generally speaking, eq. (22.72) depends on ε as on a parameter; however, since, during the course of the entire proof, ε has been considered as a fixed sufficiently small number, it follows that in order to avoid

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unnecessary complexity of the formulas, the dependence on ε is neglected here. The same procedure will be followed below, without special reservations.

It is obvious that, since the right side of eq.(22.70) is periodic and has a period of 2π with respect to the angular variable g_k, a substitution of g^0 in $T^F_{z,\,t_0}(g^0)$ by $g^0 + 2\pi$, will impart an increment 2π to the expression $\{T^F_{z}, t_0, (g^0)\}$.

Then, let F and F* be functions of the class C (D, Δ). Assume that

$$g_t = T^F_{z,\,t_0}(g^0); \quad g^*_t = T^{F^*}_{z,\,t}(g^*).$$

We have from eq.(22.70) and (22.71):

$$\left|\frac{d\,(g^*_t - g_t)}{dt}\right| = |P(t,\, g^*_t,\, F^*(t,\, g^*_t),\, \varepsilon) - P(t,\, g_t,\, F(t,\, g_t),\, \varepsilon)| \leqslant$$
$$\leqslant \lambda(\varepsilon,\; D)\|F^* - F\| + \lambda(\varepsilon,\; D)(1 + \Delta)|g^*_t - g_t|, \quad (22.73)$$

where, in general,

$$\|f\| = \sup_{t,\,g} |f(t,\; g)|.$$

However, on the other hand,

$$g^*_t - g_t = g^* - g^0 \quad \text{as} \quad t = t_0.$$

For this reason, we prove from eq.(22.73) that

$$|T^{F^*}_{z,\,t_0}(g^*) - T^F_{z,\,t_0}(g^0)| =$$
$$= |g^*_t - g_t| \leqslant g^* - g^0|\exp\{\lambda(\varepsilon,\; D)(1 + \Delta)|z|\} +$$
$$+ \frac{\|F^* - F\|}{1 + \Delta}\{\exp|\lambda(\varepsilon,\; D)(1 + \Delta)|z|| - 1\}. \quad (22.74)$$

After these preliminary remarks, let us consider the transformation S, transforming the function F of C (D, Δ) into the function

$$S_{t,\,g}(F) = \int_{-\infty}^{\infty} J(z)\,Q\,\{t + z;\; T^F_{z,\,t}(g);\; F[t + z;\; T^F_{z,\,t}(g)];\; \varepsilon\}\,dz.$$

(22.75)

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By virtue of the above property of periodicity of $T^F_{z,t}(g)$ with respect to the angular variable, we can conclude that the transformed function $S_{t,g}(F)$ likewise possesses the period of 2π with respect to g.

Let us evaluate

$$|S_{t,g}(F)|;\quad |S_{t,g^*}(F^*) - S_{t,g}(F)|.$$

We have, by virtue of conditions c), d) and of eq.(22.69),

$$|Q\{t+z;\ T^F_{z,t}(g);\ F[t+z;\ T^F_{z,t}(g)];\ \varepsilon\}| \leqslant$$
$$\leqslant |Q\{t+z;\ T^F_{z,t}(g);\ 0;\ \varepsilon\}| + |Q\{t+z;\ T^F_{z,t}(g);$$
$$F[t+z;\ T^F_{z,t}(g)];\ \varepsilon\} - Q\{t+z;\ T^F_{z,t}(g);\ 0;\ \varepsilon\}| \leqslant$$
$$\leqslant M(\varepsilon) + \lambda(\varepsilon,\ D)D,$$

Hence, on the basis of eq.(22.68), we obtain from eq.(22.75) the following:

$$|S_{t,g}(F)| \leqslant \{M(\varepsilon) + \lambda(\varepsilon,\ D)D\}\, K \int_{-\infty}^{\infty} e^{-\alpha|z|}\,dz =$$
$$= \frac{2K}{\alpha}\{M(\varepsilon) + \lambda(\varepsilon,\ D)D\}. \qquad (22.76)$$

In a completely analogous manner, we have

$$|S_{t,g^*}(F^*) - S_{t,g^0}(F)| \leqslant$$
$$\leqslant K \int_{-\infty}^{\infty} e^{-\alpha|z|}\lambda(\varepsilon,\ D)\{|T^{F^*}_{z,t}(g^*) - T^F_{z,t}(g^0)| +$$
$$+ |F^*[t+z;\ T^{F^*}_{z,t}(g^*)] - F[t+z;\ T^F_{z,t}(g^0)]|\}\,dz \leqslant$$
$$\leqslant K\lambda(\varepsilon,\ D)\|F^* - F\| \int_{-\infty}^{\infty} e^{-\alpha|z|}\,dz +$$
$$+ K\lambda(\varepsilon,\ D)(1+\Delta) \int_{-\infty}^{\infty} e^{-\alpha|z|}|T^{F^*}_{z,t}(g^*) - T^F_{z,t}(g^0)|\,dz,$$

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whence, in consequence of eq. (22.74), it follows that

$$|S_{t,\,g^*}(F^*)-S_{t,\,g}(F)|\leqslant\{|g^*-g^0|(1+\Delta)+\|F^*-F\|\}\lambda(\varepsilon,D)\,K\times$$
$$\times\int\limits_{-\infty}^{\infty}\exp\{-\alpha|z|+\lambda(1+\Delta)|z|\}\,dz. \quad (22.77)$$

Up to now, the quantities D and Δ have been arbitrary; let us now select them as functions of the parameter: $D = D(\varepsilon)$, $\Delta = \Delta(\varepsilon)$, such that $D(\varepsilon) \to 0$, $\Delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and that for all ε less than a certain ε_0, the following inequations are valid:

$$\frac{2K}{\alpha}\{M(\varepsilon)+\lambda(\varepsilon,\ D)D\}<D;\quad \frac{4K}{\alpha}\lambda(\varepsilon,\ D)(1+\Delta)<\Delta,$$
$$(1+\Delta)\lambda(\varepsilon,\ D)<\frac{\alpha}{2};\quad \frac{8\lambda(\varepsilon,\ D)}{\alpha}<1. \quad (22.78)$$

Such a selection $D = D(\varepsilon)$, $\Delta = \Delta(\varepsilon)$ is always possible since $M(\varepsilon) \to 0$, $\lambda(\varepsilon, D) \to 0$ as $\varepsilon \to 0$, $D \to 0$. For the D and Δ selected, we obtain from eqs. (22.76) and (22.77) the following inequations:

$$|S_{t,\,g}(F)|<D(\varepsilon),$$
$$|S_{t,\,g^*}(F^*)-S_{t,\,g}(F)|\leqslant\Delta(\varepsilon)|g^*-g^0|+\frac{1}{2}\|F^*-F\| \quad (22.79)$$

and, in particular,

$$|S_{t,\,g^*}(F)-S_{t,\,g^0}(F)|\leqslant\Delta(\varepsilon)|g^*-g^0|. \quad (22.80)$$

Thus, for $\varepsilon < \varepsilon_0$, the transformation S reflects the class of functions C {D (ε); D (ε)} into itself. Since, in addition, eq. (22.79) yields the inequation

$$\|SF^*-SF\|\leqslant\frac{1}{2}\|F^*-F\|, \quad (22.81)$$

the well-known result known also as the Cacciopoli-Banach principle in functional analysis, indicates directly that the equation

$$F=SF \quad (22.82)$$

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in the class of functions C (D, Δ) has a unique solution.

The existence of this solution may be established by the effective method of iteration

$$F_{p+1} = SF_p \quad (p = 1,\ 2,\ \ldots), \tag{22.83}$$

since the ration(22.81) guarantees the convergence of the sequence F_p.

Denoting this solution by

$$F = f(t,\ g,\ \varepsilon),$$

we see that, by the very definition of the class C (D (ε), Δ (ε)), the function f satisfies the conditions (22.63) of our argument.

Further, differentiating the iteration formulas (22.83) with respect to g, it will become obvious from eq.(22.81) that, if the functions P (t, g, h, ε), Q (t, g, h, ε) in the region

$$t \in R,\quad g \in \Omega,\quad h \in U_\rho,\quad 0 < \varepsilon < \varepsilon^*$$

have bounded and uniformly continuous derivatives with respect to g, h to the m^{th} order inclusive, then the derivatives F_p with respect to g to the m^{th} order inclusive will likewise be uniformly convergent.

Thus the function f (t, g, ε) satisfies the conditions of Argument I; therefore, to complete its proof, it remains for us to show that the relations

$$h = f(t,\ g,\ \varepsilon) \tag{22.84}$$

define an integral manifold for the system of differential equations (22.6) under consideration.

Expanding eq.(22.82) will give

$$f(t,\ g,\ \varepsilon) = \int_{-\infty}^{\infty} J(z)\, Q\{t+z;\ T^f_{z,\,t}(g);\ f[t+z;\ T^f_{z,\,t}(g)];\ \varepsilon\}\, dz.$$

Let us now relpace g by $T^f_{t-t_0,t_0}$ (g) and note that identically

$$T^f_{z,\,t_0} T^f_{t-t_0,\,t_0} = T^f_{z+t-t_0,\,t_0}.$$

Then, by introducing τ = z + t instead of z as the integration variable,

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and putting

$$f\{t;\ T^f_{t-t_0,t_0}(g);\ \varepsilon\} = h_t;\quad T^f_{t-t_0,t_0}(g) = g_t,$$

we obtain:

$$h_t = \int_{-\infty}^{\infty} J(\tau - t)\, Q(\tau,\ g_\tau,\ h_\tau,\ \varepsilon)\, d\tau =$$

$$= \int_{-\infty}^{t} J(\tau - t)\, Q(\tau,\ g_\tau,\ h_\tau,\ \varepsilon)\, d\tau + \int_{t}^{\infty} J(\tau - t)\, Q(\tau,\ g_\tau,\ h_\tau,\ \varepsilon)\, d\tau.$$

Thence, taking into consideration the properties of eqs.(22.66) and (22.67), and of the function J (z), we see that h_t is differentiable with respect to t, and that

$$\frac{dh_t}{dt} = Hh_t + Q(t,\ g_t,\ h_t,\ \varepsilon).$$

On the other hand, by definition of the operator $T^f_{t-t_0,t_0}$, we have

$$\frac{dg_t}{dt} = G(\varepsilon) + P(t,\ g_t,\ h_t,\ \varepsilon).$$

Hence

$$g_t = T^f_{t-t_0,t_0}(g);\quad h_t = f(t,\ g_t,\ \varepsilon)$$

represents the solution of the system of differential equations under consideration, which, for $t = t_0$, reduces to g; f $(t_0,\ g,\ \varepsilon)$. Thus it is obvious that the manifold (22.84) is integral for this system, which completes the proof of Argument I.

Argument II. If there exists a sequence of real numbers $\{\tau_m\}$ such that, for a certain $\varepsilon < \varepsilon^*$, uniformly on $R\Omega U_\rho$:

$$|P(t + \tau_m,\ g,\ h,\ \varepsilon) - P(t,\ g,\ h,\ \varepsilon)| \to 0,$$
$$|Q(t + \tau_m,\ g,\ h,\ \varepsilon) - Q(t,\ g,\ h,\ \varepsilon)| \to 0;\quad m \to \infty,$$

then, for this ε, uniformly on $R\Omega$, we will obtain

$$|f(t + \tau_m,\ g,\ \varepsilon) - f(t,\ g,\ \varepsilon)| \to 0;\quad m \to \infty.$$

Proof of Argument II. Let us take a certain function F (t, g) from the class

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$C(D, \Delta)$ where, as in the proof of the preceding argument, $D = D(\varepsilon)$, $\Delta = \Delta(\varepsilon)$.

Consider the expressions

$$y_z = T^F_{z,t}(g); \quad y^\tau_z = T^F_{z,t+\tau}(g)$$

and note that

$$\frac{dy_z}{dz} = G(z) + P\{z+t;\ y_z;\ F[z+t,\ y_z];\ \varepsilon\}, \tag{22.85}$$

$$\frac{dy^\tau_z}{dz} = G(z) + P\{z+t+\tau;\ y^\tau_z;\ F[z+t+\tau;\ y^\tau_z];\ \varepsilon\}, \tag{22.86}$$

$$y_z = g;\ y^\tau_z = g \quad \text{as} \quad z = 0.$$

Let us denote:

$$\|P_\tau - P\| = \sup_{t,g,h} |P(t+\tau, g, h, \varepsilon) - P(t, g, h, \varepsilon)|;$$

$$t \in R;\ g \in \Omega;\ h \in U_\rho,$$

$$\|F_\tau - F\| = \sup_{t,g} |F(t+\tau,\ g) - F(t,\ g)|;\ t \in R,\ g \in \Omega.$$

Subtracting eq.(22.85) from eq.(22.86), and applying the usual majoration process, we obtain

$$\left|\frac{\partial (y^\tau_z - y_z)}{\partial z}\right| \leqslant \lambda(\varepsilon,\ D)(1+\Delta)|y^\tau_z - y_z| +$$
$$+ \lambda(\varepsilon,\ D)\|F_\tau - F\| + \|P_\tau - P\|.$$

Whence, since $|y^\tau_z - y_z| = 0$, for $z = 0$, we have

$$|y^\tau_z - y_z| \leqslant \frac{\|P_\tau - P\| + \lambda(\varepsilon, D)\|F_\tau - F\|}{(1+\Delta)\lambda(\varepsilon, D)} \times$$
$$\times \{\exp[\lambda(\varepsilon,\ D)(1+\Delta)|\Delta|] - 1\}. \tag{22.87}$$

On the other hand, by definition of the transformation S, it follows that

$$S_{t+\tau, g}(F) - S_{t,g}(F) =$$
$$= \int_{-\infty}^{\infty} J(z)\{Q[t+z+\tau;\ y^\tau_z;\ F(t+z+\tau;\ y^\tau_z);\ \varepsilon] -$$
$$- Q[t+z;\ y_z;\ F(t+z,\ y_z);\ \varepsilon]\}\, dz,$$

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and therefore

$$|S_{t+\tau,g}(F)-S_{t,g}(F)|\leqslant K\int_{-\infty}^{\infty}e^{-\alpha|z|}\{\lambda(\varepsilon,D)(1+\Delta)|y_z^{\tau}-y_z|+$$
$$+\|Q_\tau-Q\|+\lambda(\varepsilon,D)\|F_\tau-F\|\}\,dz.$$

Thus, bearing in mind the inequations (22.78) and (22.87), we get

$$|S_{t+\tau,g}(F)-S_{t,g}(F)|\leqslant$$
$$\leqslant\frac{4K}{\alpha}\{\|P_\tau-P\|+\lambda(\varepsilon,D)\|F_\tau-F\|\}+\frac{2K}{\alpha}\|Q_\tau-Q\|$$

or, in abbreviated form,

$$\|(SF)_\tau-SF\|\leqslant\frac{1}{2}\|F_\tau-F\|+\frac{2K}{\alpha}\{\|Q_\tau-Q\|+2\|P_\tau-P\|\}. \quad (22.88)$$

Now let us take successively:

$$F_0=0;\quad F_1=SF_0;\ \ldots;\ F_{n+1}=SF_n;\ \ldots$$

Because of the inequation (22.81), we have

$$\|F_{n+1}-F_n\|=\|SF_n-SF_{n-1}\|\leqslant\frac{1}{2}\|F_n-F_{n-1}\|,$$

so that

$$\|F_{n+m}-F_n\|\leqslant\left(\frac{1}{2}\right)^{n-1}\|F_1\|$$

Consequently, the sequence (F_N) uniformly converges to f:

$$\|F_N-f\|\leqslant\left(\frac{1}{2}\right)^{N-1}\|F_1\|\to 0\qquad N\to\infty. \quad (22.89)$$

Put, for brevity,

$$\sigma_\tau=\frac{2K}{\alpha}\{\|Q_\tau-Q\|+2\|P_\tau-P\|\}. \quad (22.90)$$

Then, from eq.(22.88) we obtain successively:

$$\|(F_1)_\tau-F_1\|\leqslant\sigma_\tau;\ \ \|(F_2)_\tau-F_2\|\leqslant$$

$$\leqslant \left(1+\frac{1}{2}\right)\sigma_\tau; \quad \|(F_3)_\tau - F_3\| \leqslant \left(1+\frac{1}{2}+\frac{1}{4}\right)\sigma_\tau; \quad \ldots$$

In general,

$$\|(F_N)_\tau - F_N\| \leqslant 2\sigma_\tau,$$

whence, passing to the limit as $N \to \infty$, we convince ourselves on the basis of eq. (22.89) that

$$\|(f)_\tau - f\| \leqslant 2\sigma_\tau$$

or, expanding and remembering eq. (22.90), we see that everywhere on $R\Omega$, for any τ, the inequality

$$|f(t+\tau, g, \varepsilon) - f(t, g, \varepsilon)| \leqslant \frac{4K}{\sigma}\{\|Q_\tau - Q\| + 2\|P_\tau - P\|\}.$$

will be valid.

Now let $\{\tau_m\}$ be a sequence out of R such that for a certain $\varepsilon < \varepsilon_o$, uniformly on $R\Omega$:

$$|Q(t+\tau_m, g, h, \varepsilon) - Q(t, g, h, \varepsilon)| \to 0;$$
$$|P(t+\tau_m, g, h, \varepsilon) - P(t, g, h, \varepsilon)| \to 0.$$

Then, for this value of ε we have

$$\|Q_{\tau_m} - Q\| \to 0; \quad \|P_{\tau_m} - P\| \to \infty; \quad m \to \infty,$$

and, consequently, the relation

$$|f(t+\tau_m, g, \varepsilon) - f(t, g, \varepsilon)| \to 0; \quad m \to \infty,$$

uniformly holds on $R\Omega$, thus proving Argument II.

Argument III. It is possible to assign positive constants ε_1, γ, C, σ_o, σ_1 ($\sigma_o < \sigma_1 < \rho$), such that, if s eigenvalues of a matrix have negative real parts, and the remaining ones have positive real parts, then, for every $\varepsilon < \varepsilon_1$ of any real t_o and any g_o of Ω in the neighborhood U_{σ_o}, there exists an s-dimensional manifold $\mathfrak{M}(t_o, g_o, \varepsilon)$ of points of $\{h\}$ with the properties:

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1) If* for $t = t_0$

$$h_t \in U_{\sigma_0}; \quad h_t \overline{\in} \mathfrak{M}(t, g_t, \varepsilon),$$

then, for a certain $\bar{t} > t_0$,

$$h_{\bar{t}} \overline{\in} U_{\sigma_1}.$$

2) If for $t = t_0$,

$$h_t \in \mathfrak{M}(t, g_t, \varepsilon),$$

then, for all $t \geq t_0$,

$$|h_t - f(t, g_t, \varepsilon)| \leqslant Ce^{-\gamma(t-t_0)} |h_0 - f(t_0, g_0, \varepsilon)|,$$

where (g_0, h_0) represent (g_t, h_t) for $t = t_0$.

3) If all eigenvalues of the matrix H have positive real parts $(s = 0)$, then the manifold $\mathfrak{M}(t_0, g_0, \varepsilon)$ degenerates to the point: $h = f(t_0, g_0, \varepsilon)$.

4) If, on the other hand, the real parts of all eigenvalues of the matrix H are negative, then the manifold (t_0, g_0, ε) coincides with the whole neighborhood U_{σ_0}.

Proof of Argument III. Consider the integral-differential system:

$$\left.\begin{aligned} h_t &= \int_{t_0}^{\infty} J(\tau - t) Q(\tau, g_\tau, h_\tau, \varepsilon)\, d\tau + J(t_0 - t) A; \ t_0 < t, \\ \frac{dg_t}{dt} &= G(\varepsilon) + P(t, g_t, h_t, \varepsilon); \quad t_0 < t, \quad g_t = g_0; \quad t = t_0, \end{aligned}\right\} \qquad (22.91)$$

where A is a certain arbitrary fixed vector of E_{n-1}.

Using, for the investigation of this system, the methods of obtaining evaluations that have been repeatedly employed above, it is not difficult to establish the following result:

It is possible to assign positive ε_1, σ_0, σ, obeying the conditions

* Here, as usual, $g = g_t$, $h = h_t$ denotes a solution of the system of differential equations (22.61) under consideration.

$$\varepsilon_1 < \varepsilon_0; \quad D(\varepsilon_1) < \sigma_0; \quad \sigma_0 < \sigma_1 < \rho, \tag{22.92}$$

such that, for any value of the parameter ε and any vector A of E_{n-1}, satisfying the inequalities

$$0 < \varepsilon < \varepsilon_1; \quad |A| < \sigma_0,$$

a) the system (22.91) will have a unique solution (g_t, h_t), for which $h_t \in U_{\sigma_1}$ for all $t \geq t_0$;

b) for this solution

$$h_t = \Psi(t_0, t, g_t, A, \varepsilon), \tag{22.93}$$

where $\Psi(t_0, t, g, A, \varepsilon)$ is a continuous function of its arguments, satisfying a Lipschitz inequality of the form

$$\begin{aligned} &|\Psi(t_0, t, g', A', \varepsilon) - \Psi(t_0, t, g'', A'', \varepsilon)| \leq \\ &\leq \nu(\varepsilon, \sigma_0)|g' - g''| + \mu(\varepsilon, \sigma_0)e^{-\frac{\gamma}{2}(t-t_0)}|A' - A''|; \quad t \geq t_0, \end{aligned} \tag{22.94}$$

where $\nu(\varepsilon, \sigma_0)$ and $\mu(\varepsilon, \sigma_0)$ tend toward zero as $\varepsilon \to 0$, $\sigma_0 \to 0$.

Moreover, eq.(22.66) shows that the solutions of the integral-differential system are at the same time also solutions of the differential equations (22.61).

On the other hand, let (g_t, h_t) be any solution of eq.(22.61), for which $h_0 \in U_{\sigma_0}$, and $h_t \in U_{\sigma_1}$ for all $t > t_0$.

For brevity let us call such solutions type S solutions. We have

$$\int_t^\infty J(\tau - t)\frac{dh_\tau}{d\tau}d\tau =$$

$$= \int_t^\infty J(\tau - t)Hh_\tau d\tau + \int_t^\infty J(\tau - t)Q(\tau, g_\tau, h_\tau, \varepsilon)d\tau, \tag{22.95}$$

$$\int_{t_0}^t J(\tau - t)\frac{dh_\tau}{d\tau}d\tau =$$

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$$-\int_{t_0}^{t} J(\tau-t)Hh_\tau\,d\tau+\int_{t_0}^{t} J(\tau-t)Q(\tau,\ g_\tau,\ h_\tau,\ \varepsilon)\,d\tau. \qquad (22.96)$$

Integrating term-by-term and taking eq.(22.66) into consideration, we obtain

$$\int_{t}^{\infty} J(\tau-t)\frac{dh_\tau}{d\tau}\,d\tau=-J(+0)\,h_t-\int_{t}^{\infty}\frac{dJ(\tau-t)}{d\tau}\,h_\tau\,d\tau=$$

$$=-J(+0)\,h_t+\int_{t}^{\infty} J(\tau-t)\,Hh_\tau\,d\tau,$$

$$\int_{t_0}^{t} J(\tau-t)\frac{dh_\tau}{d\tau}\,d\tau=$$

$$=J(-0)\,h_t-J(t_0-t)\,h_0+\int_{t_0}^{t} J(\tau-t)\,Hh_\tau\,d\tau.$$

Thus, by adding eqs.(22.95) and (22.96), and noting that $J(-0)\,h_t-J(+0)\,h_t=h_t$, we can prove that

$$h_t=\int_{t_0}^{\infty} J(\tau-t)\,Q(\tau,\ g_\tau,\ h_\tau,\ \varepsilon)\,d\tau+J(t_0-t)\,h_0.$$

Consequently, each type S solution is a solution of the integral-differential system (22.99) for $A=h_0$, so that we may write the following expression for it

$$h_t=\Psi(t_0,\ t,\ g_t,\ A,\ \varepsilon);\qquad |A|<\sigma_0. \qquad (22.97)$$

Moreover, by virtue of the Argument I and the inequations (22.63), (22.92), all the solutions lying on the integral manifold

$$h=f(t,\ g,\ \varepsilon),$$

belong to the S type, so that for each of them a corresponding $A=A'$ can be indicated. For this reason, in view of eq.(22.94), we have

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$$|f(t, g, \varepsilon) - \Psi(t_0, t, g, A, \varepsilon)| \leqslant$$
$$\leqslant \mu(\varepsilon, \sigma_0) e^{-\frac{\alpha}{2}(t-t_0)} |A' - A|; \quad t \geqslant t_0.$$

In particular, for each type S solution, by replacing the arbitrary g by g_t, we obtain

$$|f(t, g_t, \varepsilon) - h_t| \leqslant \mu(\varepsilon, \sigma_0) e^{-\frac{\alpha}{2}(t-t_0)} |f(t_0, g_0, \varepsilon) - h_0|. \qquad (22.98)$$

Consider now the point set $\{h\}$ belonging to U_{σ_0}, for which

$$h = \Psi(t_0, t_0, g_0, A, \varepsilon); \quad |A| < \sigma,$$

corresponding to the given fixed t_0, g_0, ε, and let us denote it by $\mathfrak{M}(t_0, g_0, \varepsilon)$.

Since, for any type S solution, the relation (22.97) will be satisfied, it is possible when assuming that $t = t_0$ to prove that h_0 must belong to $\mathfrak{M}(t_0, g_0, \varepsilon)$. Consequently, if for $t = t_0$ we have

$$h_t \in U_{\sigma_1}; \quad h_t \notin \mathfrak{M}(t, g_t, \varepsilon),$$

then the solution (g_t, h_t) corresponding to these initial conditions cannot belong to the S type, so that h_t cannot remain in the neighborhood of U_{σ_1} for all $t > t_0$.

As already remarked, the solution of the integral-differential system (22.91), having the properties a), b), the existence of which has been established above, is at the same time a solution of the differential equations (22.61). Owing to property b) we have:

$$h_0 = \Psi(t_0, t_0, g_0, A, \varepsilon), \qquad (22.99)$$

Since the solution of eq.(22.61) is entirely determined by the initial conditions, it is obvious that if (g_t, h_t) is some solution of eq.(22.61), for which eq.(22.99) holds, then it is also a solution of the integral-differential system (22.99) and possesses the properties a), b).

Thus we may assert that if, for a certain solution of eq.(22.61), at $t = t_0$,

$$h_t \in \mathfrak{M}(t, g_t, \varepsilon),$$

then it will belong to the S type, and therefore, the inequation (22.98) will hold for it. Having now established the required properties 1), 2) of the manifold $\mathfrak{M}(t, g_t, \varepsilon)$, we will still prove the properties 3), 4), thereby completing our

proof of Argument III.

Now let s = 0. Then, by definition in eq.(22.65)

$$J(t) = -e^{-Ht}, \quad t > 0; \qquad J(t) = 0, \quad t < 0,$$

as a result of which the integral equation of system (22.91) takes on the form

$$h_t = -\int_t^{\infty} e^{H(t-\tau)} Q(\tau, g_\tau, h_\tau, \varepsilon)\, d\tau, \quad t \geq t_0,$$

in which the arbitrary vector A does not figure. Thence it is clear that $\mathfrak{M}(t_o, g_o, \varepsilon)$ degenerates into a single point; since $f(t_o, g_o, \varepsilon)\, \mathfrak{M}(t_o, g_o, \varepsilon)$ is always true, we see that in this case $\mathfrak{M}(t_o, g_o, \varepsilon)$ consists of a single point, which is $f(t_o, g_o, \varepsilon)$.

Let now, on the other hand, s = n − 1. Then, by definition (22.65)

$$J(t) = 0, \quad t > 0; \quad J(t) = e^{Ht}, \quad t < 0,$$

and the integral equation of system (22.91) will be:

$$h_t = \int_{t_0}^{t} e^{H(t-\tau)} Q(\tau, g_\tau, h_\tau, \varepsilon)\, d\tau + e^{H(t-t_0)} A, \quad t \geq t_0, \qquad (22.100)$$

whence, more specifically, it follows that $A = h_o$.

It is not difficult to convince ourselves that eq.(22.100) is an identity for any solution of the differential equations (22.61) for any h_o. Thus, in this case,

$$\mathfrak{M}(t_0, g_0, \varepsilon) = U_{s}.$$

Finally, let s and n − 1 − s be both different from zero. In this case, the term $J(t_o - t)\, A$, by means of which the vector A enters the integral-differential system (22.91), may be represented in the form

$$U \begin{vmatrix} 0 & 0 \\ 0 & e^{H_-(t-t_0)} \end{vmatrix} U^{-1} A = U \begin{vmatrix} 0 & 0 \\ 0 & e^{H_-(t-t_0)} \end{vmatrix} U^{-1} a,$$

where

$$a = U \begin{vmatrix} 0 & 0 \\ 0 & 1_s \end{vmatrix} U^{-1} A \qquad (22.101)$$

and 1_s is the s-dimensional unit matrix.

Hence it may be concluded that, identically,

$$\Psi(t_0, t, g, A, \varepsilon) = \Psi(t_0, t, g, a, \varepsilon).$$

On the other hand, with an arbitrary A, the vector a, defined by eq.(22.101), always has s independent components $a_1, \ldots, a_s$, in consequence of which the equation

$$h = \Psi(t_0, t_0, g_0, A, \varepsilon),$$

characterizing the manifold $\mathfrak{M}(t_o, g_o, \varepsilon)$ may be represented in the form

$$h = h(a_1, \ldots, a_s),$$

where $h(a_1, \ldots, a_s)$ are functions of s-parameters depending on t_o, g_o, ε and, by virtue of (22.94), satisfy the Lipschitz conditions.

Thus $\mathfrak{M}(t_o, g_o, \varepsilon)$ is an s-dimensional manifold. This proves Argument III.

Remark. It follows from Argument III that, in the neighborhood of ΩU_{σ_o} there can be only one unique integral manifold for the system (22.61), namely the manifold (22.62).

Indeed, this assertion is obvious in the case $s = 0$. The case $s = n - 1$ passes into the first case if, in the equations, t is replaced by $-t$.

The case of $0 < s < n - 1$ remains to be considered. Let a certain integral manifold S_t lie in the neighborhood ΩU_{σ_o}.

$$S_t \in \Omega U_{\sigma_0}; \quad -\infty < t < \infty.$$

Then it follows from Argument III that, if $(g_o, h_o) \in S_t$, then it must also be true that

$$h_0 \in \mathfrak{M}(t_0, g_0, \varepsilon).$$

Taking arbitrarily the small positive η, let us select a positive z satisfying the inequation

$$2\sigma_0 C e^{-\gamma z} < \eta;$$

let us take the arbitrary real t_1 and put $t_o = t_1 - z$. Then,

$$2\sigma_0 C e^{-\gamma(t_1 - t_0)} < \eta.$$

Let, on the other hand, (g, h) be an arbitrary point S_{t_1}. We note that by the definition of the integral manifold, the solution (g_t, h_t) of the system (22.61),

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which takes on the value (g, h) as $t = t_1$, lies on S_t for any t. We have, in particular, $(g_{t_0}, h_{t_0}) \in S_{t_0}$, so that

$$h_{t_0} \in \mathfrak{M}(t_0, g_{t_0}, \varepsilon).$$

However, according to Argument III, we then have

$$|h - f(t_1, g, \varepsilon)| = |h_{t_1} - f(t_1, g_{t_1}, \varepsilon)| \leqslant$$
$$\leqslant Ce^{-\gamma(t_1-t_0)}|h_{t_0} - f(t_0, g_{t_0}, \varepsilon)| < 2\varepsilon_0 Ce^{-\gamma(t_1-t_0)} < \eta,$$

whence, because of the arbitrariness of η:

$$h = f(t_1, g, \varepsilon). \tag{22.102}$$

Thus, the relation $(g, h) \in S_{t_1}$ gives eq. (22.102), which proves our proposition.

We now introduce certain definitions relating to the theory of quasi-periodic functions.

Consider some function f(t, x) assigned on RE, where E is a certain set of values of x.

We will postulate that f(t, x) is a quasi-periodic function of t, uniformly varying with respect to x, for the case that $\eta > 0$ can be associated with a positive $l(\eta)$ such that, any interval on R of the length $l(\eta)$ contains at least one τ (almost a period for η) for which the inequation

$$|f(t+\tau, x) - f(t, x)| \leqslant \eta.$$

is everywhere true along RE.

For these almost periodic functions, a countable set of frequencies $\{\lambda_j\}$ exists, not depending on x, so that, for any λ not belonging to it, we have

$$\lim \frac{1}{T}\int_0^T f(t, x)e^{-i\lambda t}\,dt = 0.$$

Let $\{\omega_\alpha\}$ represent a countable set of real numbers possessing the following properties:

1) Between the ω_α there exist no nontrivial linear relations

$$\sum n_\alpha \omega_\alpha = 0$$

with integer coefficients.

2) Every λ_j can be represented by a linear combination ω_α with coefficients

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that are integers.

Such a set $\{\omega_\alpha\}$ will be called the frequency base of the given quasi-periodic function. In particular, for a periodic function, the frequency base consists of a single element; for quasi-periodic functions, the frequency base consists of a finite number of elements.

It is well known that a frequency base possesses the following important property: If τ_m is a sequence such that, for any ω_α, we have

$$e^{i\omega_\alpha \tau_m} \underset{m\to\infty}{\longrightarrow} 1, \tag{22.103}$$

then, uniformly on RE,

$$f(t+\tau_m, x) - f(t, x) \underset{m\to\infty}{\rightarrow} 0. \tag{22.104}$$

This property may also serve as a definition for the quasi-periodic functions under consideration.

Namely, if $\{\omega_\alpha\}$ is a countable set of linearly independent real numbers and if, for each sequence τ_m for which eq.(22.103) is true, the relation (22.104) will also be true uniformly on RE, then $f(t,x)$ will be almost periodic uniformly with respect to x, and $\{\omega_\alpha\}$ will be its frequency base.

Noting this, we recall that, in our preceding reasoning, we have encountered assertions of the following type:

If $\{\tau_m\}$ is a sequence for which, uniformly on RE,

$$f(t+\tau_m, x) - f(t, x) \underset{m\to\infty}{\rightarrow} 0,$$

then, uniformly on RE,

$$F(t+\tau_m, x) - F(t, x) \underset{m\to\infty}{\rightarrow} 0.$$

We may then assert that in cases where $f(t, x)$ is a function that is quasi-periodically uniform with respect to x, and the set $\{\omega_\alpha\}$ is its frequency base, then $F(t, x)$ will likewise be a function quasi-periodically uniform with respect to x, having the same frequency base.

Section 23. Periodic and Quasi-Periodic Solutions

Let us now pass to an application of the arguments proved in the preceding

Section.

These arguments were formulated with respect to the system (22.61) to which the the system (22.24) was reduced, to which, in turn, under suitable conditions, the fundamental equation (21.1) was reduced.

Here it only remains to transfer the formulation of the properties of the solutions to the solution of this fundamental equation.

Let us begin with the case of a quasi-static solution, when the dependence on the angular variable is canceled.

Taking Arguments I, II, III and the corresponding remarks into consideration, we arrive directly at the following theorem:

Theorem I. Let the function $X(t, x)$ entering into the equation

$$\frac{dx}{dt} = \varepsilon X(t, x), \tag{23.1}$$

satisfy the following conditions for values of x located in a certain region E:

a) $X(t, x)$ is a quasi-periodic function of t uniform with respect to x;

b) $X(t, x)$ and its partial derivatives of the first order, with respect to x, are bounded and uniformly continuous for

$$-\infty < t < \infty, \quad x \in E.$$

Let, further, the equation of first approximation

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi), \tag{23.2}$$

in which

$$X_0(\xi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(t, \xi)\, dt \tag{23.3}$$

has the quasi-static solution ξ^0 in the region E, with the real parts of all roots of the characteristic equation

$$\mathrm{Det}\,|pI - X_0'(\xi^0)| = 0 \tag{23.4}$$

being different from zero.

Then positive ε_0, σ_0, $\sigma_1 (\sigma_0 < \sigma_1)$ may be assigned such that, for every positive ε less than ε_0 the following assertions are true:

1) Equation (23.1) has a unique solution $x = x^*(t)$, determinate over the en-

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tire interval $(-\infty, \infty)$, for which everywhere

$$|x^*(t) - \xi^0| < \sigma_0.$$

2) This solution $x^*(t)$ is quasi-periodic, having the frequency base of the function $X(t, x)$.

3) $|x^*(t) - \xi^0| \leqslant \delta(\varepsilon) \underset{\varepsilon \to 0}{\to} 0 \quad (-\infty < t < \infty)$.

4) Let $x = x(t)$ represent any solution of eq. (23.1) different from $x^*(t)$, which for some t_0 satisfies an inequation of the form

$$|x(t_0) - \xi^0| < \sigma_0.$$

Then, if the real parts of all roots of the characteristic equation (23.4) are negative, the distance $|x(t) - x^*(t)|$ tends toward zero as $t \to +\infty$, and

$$|x(t) - x^*(t)| \leqslant Ce^{-\gamma(t-t_0)}; \ C, \ \gamma = \text{const}, \ \gamma > 0. \tag{23.5}$$

If the real parts of all these roots are positive, then a $t_1 > t_0$ can be found, such that

$$|x(t_1) - \xi^0| > \sigma_1. \tag{23.6}$$

If s real parts of these roots are negative, and the remaining $n - s$ real parts are positive, then in the σ_0-neighborhood of the point ξ_0 there exists an s-dimensional manifold $\mathfrak{M}_{t_0}$ such that, from the relation

$$x(t_0) \in \mathfrak{M}_{t_0}$$

it follows that the difference $x(t) - x^*(t)$ of eq. (23.5) tends exponentially toward zero, and from the relation

$$x(t_0) \in \mathfrak{M}_{t_0}$$

it follows that the inequation (23.6) is true.

This theorem will be further discussed below.

It is clear that, by virtue of the property 4), the solution $x^*(t)$ will be stable when the real parts of all roots of the characteristic equation under consideration are negative.

If the real part of even one of these roots is positive, then the solution is

unstable.

Let us now consider the case when $X(t, x)$ is a periodic function of t with a certain period τ not depending on x.

Then, in particular, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T X(t, x)\,dt = \frac{1}{\tau} \int_0^\tau X(t, x)\,dt,$$

and, therefore,

$$X_0(\xi) = \frac{1}{\tau} \int_0^\tau X(t, \xi)\,dt. \tag{23.7}$$

Since the frequency base of the function $X(t, x)$ consists of a single number $\frac{2\pi}{\tau}$, the property 2) indicates that, in this case, the solution $x^*(t)$ will be periodic, with the period τ.

We note, finally, that all of the results formulated above are directly applicable to this case as well, when the fundamental equation has a somewhat more general form:

$$\frac{dx}{dt} = \varepsilon X(t, x, \varepsilon), \tag{23.8}$$

Then, in the equation of first approximation, we put

$$X_0(\xi) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau X(t, x, 0)\,dt. \tag{23.9}$$

It is here sufficient that the conditions imposed on $X(t, x)$ are satisfied by the function $X(t, x, \varepsilon)$, uniform with respect to ε over a certain interval $0 < \varepsilon < < \varepsilon_1$.

In this case, eq.(23.8) is reduced, by the same substitutions of variables, to a system of the form of eq.(22.61) with all of its enumerated properties.

We pass now to a consideration of the properties in the neighborhood of a periodic solution, of equations of first approximation, when the role of the angular variable is of substantial importance.

We will first state a number of consequences of Arguments I, II, III.

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Thus, it is clear from eq.(22.61) that the angular variable g, for solutions lying on the manifold

$$h = f(t,\ g,\ \varepsilon),$$

satisfies the equation

$$\frac{dg}{dt} = G(\varepsilon) + F(t,\ g,\ \varepsilon),$$

in which

$$F(t,\ g,\ \varepsilon) = P(t,\ g,\ h(t,\ g,\ \varepsilon),\ \varepsilon).$$

Here, on the basis of Argument I,
$$|F(t,\ g,\ \varepsilon)| \leqslant M(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

$$|F(t,\ g',\ \varepsilon) - F(t,\ g'',\ \varepsilon)| \leqslant N(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

We note further that F(t, g, ε) has continuous derivatives with respect to g to the m^{th} order inclusive, if the corresponding conditions in Argument I are satisfied.

It follows from Argument II that, when P, Q are quasi-periodic functions of t, uniform with respect to g, h, then F(t, g, ε) is also a quasi-periodic function of t uniform with respect to g, with the frequency base of the functions P, Q.

Now taking into consideration Argument III, we see that if the real part of even one of the roots of the equation

$$\mathrm{Det}\,|pI - H| = 0 \tag{23.10}$$

is positive, then the integral manifold under consideration will have the property of repulsion of all solutions close to it except the solutions lying on the singular manifold $\mathfrak{M}$, whose dimensionality is less than the dimensionality of the entire phase space.

For this reason, in this case, any solution lying on the integral manifold

$$h = f(t,\ g,\ \varepsilon),$$

will be unstable.

Conversely, if the real parts of all roots of eq.(23.10) are negative, then this manifold has the property of attracting the nearby solutions:

$$|h_t - f(t,\ g_t,\ \varepsilon)| \underset{t \geqslant t_0}{\leqslant} C\,|h_{t_0} - f(t_0,\ g_{t_0},\ \varepsilon)|\, e^{-\gamma(t-t_0)}.$$

From this follows, in particular, that

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$$\left|\frac{dg_t}{dt} - G(\varepsilon) - F(t, g_t, \varepsilon)\right| \underset{t > t_0}{\leqslant} C\lambda(\varepsilon, \sigma)\, e^{-\gamma(t-t_0)} \left|h_{t_0} - f(t_0, g_{t_0}, \varepsilon)\right|.$$

Summarizing the results so obtained, let us formulate them with respect to the solutions of the fundamental equation in the standard form. We recall, in this connection, that a substitution of the independent variable $\varepsilon t \to t$ was made during the process of reducing it to the form of the system (22.61).

We obtain the following theorem:

Theorem II. Let the following conditions be satisfied:

a) The equation of first approximation

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi),$$

where

$$X_0(\xi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t, \xi)\, dt,$$

has the periodic solution:

$$\xi = \xi(\varepsilon\omega t); \quad \xi(\varphi + 2\pi) = \xi(\varphi). \tag{23.11}$$

b) The real parts of all $(n - 1)$* characteristic exponents for the equations of variation

$$\frac{d\delta\xi}{dt} = \varepsilon X_0'(\xi(\varepsilon\omega t))\,\delta\xi \tag{23.12}$$

differ from zero.

c) It is possible to find a ρ-neighborhood of U_ρ of the orbit of this periodic solution, such that the function $X(t, x)$ and its partial derivatives with respect to x, to the m^{th} order inclusive, will be bounded and uniformly continuous with respect to x in the region

$$-\infty < t < \infty, \ x \in U_\rho.$$

d) $X(t \ \ x)$ is a quasi-periodic function of t, uniform with respect to $x \in U_\rho$.

Then it is possible to assign positive numbers ε_0 such that for every positive $\sigma_0(\sigma_0 < \rho)$ the following assertions are valid:

* One of the characteristic exponents, as stated above, always vanishes.

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1) The equation

$$\frac{dx}{dt} = \varepsilon X(t, x)$$

has a unique integral manifold S lying, for all real values of t, in the region U_0.

2) This manifold S allows a parametric representation of the form

$$x = f(t, \theta). \tag{23.13}$$

Here* $f(t, \theta)$ is determinate for all real t, θ, possesses the period 2π with respect to θ, and is a quasi-periodic function of t uniform with respect to θ and having the frequency base of the function $X(t, x)$. It is possible to find a function $\delta(\varepsilon)$, tending toward zero with ε, such that

$$|f(t, \theta) - \xi(\theta)| \leqslant \delta(\varepsilon). \tag{23.14}$$

The function $f(t, \theta)$ has uniformly continuous derivatives with respect to θ to the m^{th} order inclusive.

3) On the manifold S, eq.(23.1), equivalent to the equation

$$\frac{d\theta}{dt} = \varepsilon F(t, \theta), \tag{23.15}$$

in which $F(t, \theta)$ is determinate for every real t, θ, is a periodic function of θ with a period of 2π, and is a quasi-periodic function of t uniform with respect to θ and having the frequency base of the function $X(t, x)$; $F(t, \theta)$ has bounded and uniformly continuous derivatives with respect to θ, to the m^{th} order inclusive.

In addition, the following inequations are obtained:

$$\left.\begin{aligned} &|F(t, \theta) - \Omega(\varepsilon)| \leqslant \delta^*(\varepsilon), \\ &|F(t, \theta') - F(t, \theta'')| \leqslant \eta_1^*(\varepsilon)|\theta' - \theta''|, \end{aligned}\right\} \tag{23.16}$$

where

$$\delta^*(\varepsilon) \to 0, \ \eta_1^*(\varepsilon) \to 0 \ \text{ at } \ \varepsilon \to 0.$$

4) If the real parts of all n − 1 characteristic exponents considered are negative, then the manifold S has the property of attracting all solutions close to it.

Thus, let $x = x(t)$ be a solution of eq.(23.1) passing, for a certain $t = t_0$,

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* To shorten the formulas we will disregard, in the function $f(t, \theta)$ and other functions analogous to it, their explicit dependence on ε.

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through some point of the region U_{σ_0}. Then, for that solution, at $t > t_0$, inequations of the following form will be satisfied:

$$\left.\begin{aligned} |x(t) - f(t, \theta(t))| &\leqslant C_1(\varepsilon)\, e^{-\eta(t-t_0)}, \\ \left|\frac{d\theta(t)}{dt} - \varepsilon F(t, \theta(t))\right| &\leqslant C_2(\varepsilon)\, e^{-\eta(t-t_0)}. \end{aligned}\right\} \tag{23.17}$$

5) If even a single one of the real parts of the characteristic exponents considered is positive, then the manifold S is unstable. Any solution x(t), not belonging to that manifold, for which $x(t_0)$ lies in the region U_σ but not on a certain singular manifold $\mathfrak{M}_{t_0}$ of lower dimensionality, will, with the passage of time, leave the region U_σ.

With respect to this theorem, we are particularly interested in the important special case when $f(t, \theta)$ and $F(t, \theta)$ are periodic functions of t with a certain period T (not depending on θ), and the number of derivatives m is taken as equal to two.

In accordance with the property 2), such a "case of periodicity" will occur, for example, when the function X(t, x) possesses this period T with respect to t. In this case, we select the system (22.58) as the system (22.61), and therefore, the inequation (23.16) of theorem II will read

$$\Omega(\varepsilon) = \omega. \tag{23.18}$$

However, as mentioned previously, the case of periodicity will also occur if, in eq.(22.24), the functions $W(t, \varphi, b)$, $B(t, \varphi, b)$ have the form of eq.(22.59):

$$W(t, \varphi, b) = \overline{W}(t, \varphi + \nu t, b),$$
$$B(t, \varphi, b) = \overline{B}(t, \varphi + \nu t, b),$$

where $\overline{W}(t, \varphi, b)$, $\overline{B}(t, \varphi, b)$ have the period T with respect to t. Then, as the system (22.61), we take the system (22.60), whose right sides are periodic functions of the independent variable. In this case, in the inequations (23.16), theorem II will be:

$$\Omega(\varepsilon) = \frac{\nu}{\varepsilon} + \omega. \tag{23.19}$$

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The case of periodicity is of particular interest in view of the fact that here the classical results of Poincaré, as supplemented by Denjoy, can be used for analyzing the structure of the solutions lying on the manifold S.

This point will be further discussed here.

Let us take the equation

$$\frac{d\theta}{dt} = \varepsilon F(t, \theta)$$

and consider θ(t) as a function of the initial values t_o, $\theta(t_o)$ and of the difference $t - t_o$:

$$\theta(t) = \theta(t_0) + \Phi(t - t_0, t_0, \theta(t_0)). \tag{23.20}$$

We note that, by virtue of the properties of periodicity of the function $F(t,\theta)$, the function $\Phi(\tau, t_o, \theta_o)$ will be periodic with respect to the arguments t_o, θ_o with the respective periods T and 2π.

Let now

$$\theta_n = \theta(t_0 + nT).$$

We have:

$$\left.\begin{aligned} \theta_{n+1} = \theta_n + \Phi(T, t_0 + nT, \theta_n) = \theta_n + \Phi(\theta_n), \\ \Phi(\theta_n) = \Phi(T, t_0, \theta_n), \end{aligned}\right\} \tag{23.21}$$

$\Phi(\theta)$ being a periodic function with a period 2π. In view of the fact that we took m = 2, the conditions of theorem II will cause this function to have continuous derivatives of the first and second order. Further, it follows from the inequations (23.16) that

$$|\Phi'(\theta)| < \varepsilon\rho(\varepsilon); \qquad \rho(\varepsilon) \underset{\varepsilon \to 0}{\to} 0, \tag{23.22}$$

Therefore, taking the sufficiently small values of ε into consideration, for which

$$\varepsilon\rho(\varepsilon) < 1,$$

we will have

$$1 + \Phi'(\theta) > 0. \tag{23.23}$$

Thus the function

$$F(\theta) = \theta + \Phi(\theta) \tag{23.24}$$

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is monotonously increasing and has the property of periodicity "of the second kind":

$$F(\theta + 2\pi) = F(\theta) + 2\pi.$$

For this reason, the transformation

$$\theta \to F(\theta)$$

may be regarded as the mutually single-valued and mutually continuous mapping of a circle onto itself.

Equation (23.21) indicates that the successive values of the solution of equation (23.15) at the points $t = t_o + nT$ are obtained by iteration of this transformation, starting from the initial value θ_o.

We note that the iteration of transformations of the type here considered was investigated by Poincaré (Bibl.35) and Denjoy (Bibl.55), in which the following was established:

1) There exists a limit

$$\bar{\nu} = \lim_{n \to \infty} \frac{\theta_n}{2\pi n}, \tag{23.25}$$

independent of θ_o.

2) If $\bar{\nu}$ is irrational, the general solution of the iteration equation

$$\theta_{n+1} = F(\theta_n) \tag{23.26}$$

has the form

$$\theta_n = 2\pi\bar{\nu}n + \psi + E(2\pi\bar{\nu}n + \psi), \tag{23.27}$$

where ψ is an arbitrary constant. Here $E(\varphi)$ is a continuous periodic function with the period 2π. The expression $\varphi + E(\varphi)$ is a monotonously increasing function, which is not constant in any interval, no matter how small.

3) Let $\bar{\nu}$ be rational:

$$\bar{\nu} = \frac{r}{s},$$

where r, s are relatively prime numbers.

Then the iteration equation under consideration has periodic solutions for which

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$$\theta_{n+s} - \theta_n = 2\pi r. \tag{23.28}$$

Any solution θ_n, as n increases without limit, will approach one of these periodic solutions.

Let θ_m be some solution of the equation being considered, starting from an initial value θ_0, located within the interval $(0, 2\pi)$.

Then, for it, α_m, β_m

$$\alpha_m > 0, \quad \beta_m > 0, \quad \alpha_m + \beta_m < 2\pi, \tag{23.29}$$

may be assigned such that

$$-\alpha_m \leqslant \theta_{ms} - 2\pi m r \leqslant \beta_m. \tag{23.30}$$

Let us now establish several consequences of these results. Let us first consider the case of an irrational $\bar{\nu}$.

It follows from eq.(23.20) that

$$\theta(t) = \theta_n + \Phi(t - t_0 - nT, \ t_0, \ \theta_n),$$

whence

$$\theta(t) = 2\pi\bar{\nu}n + \psi + E(2\pi\bar{\nu}n + \psi) + \\ + \Phi(t - t_0 - nT, \ t_0, \ 2\pi\bar{\nu}n + \psi + E(2\pi\bar{\nu}n + \psi)).$$

Putting, for brevity,

$$\xi + E(\varphi) + \Phi\left(\frac{T}{2\pi\bar{\nu}}\xi, \ t_0, \ \varphi + E(\varphi)\right) = f(\xi, \ \varphi),$$

we may also write

$$\theta(t) = 2\pi\frac{\bar{\nu}(t - t_0)}{T} + \psi + f\left(2\pi\bar{\nu}\frac{t - t_0 - nT}{T}, \ 2\pi\bar{\nu}n + \psi\right). \tag{23.31}$$

We should mention here that the function $f(\xi, \varphi)$, so introduced, is continuous and has the period 2π with respect to φ.

Moreover, since the relation (23.31) is true for any n, it follows that

$$f\left(2\pi\bar{\nu}\frac{t - t_0 - nT}{T} - 2\pi\bar{\nu}, \ 2\pi\bar{\nu}n + \psi + 2\pi\bar{\nu}\right) = \\ = f\left(2\pi\bar{\nu}\frac{t - t_0 - nT}{T}, \ 2\pi\bar{\nu}n + \psi\right).$$

The quantity t here is also arbitrary. Let us take

$$t - t_0 - nT = \frac{T}{2\pi\bar{\nu}}\, u$$

and represent this identity in the form

$$f(u - 2\pi\bar{\nu},\quad 2\pi\bar{\nu}n + \psi + 2\pi\bar{\nu}) = f(u,\ 2\pi\bar{\nu}n + \psi).$$

However, since the numbers $2\pi\bar{\nu}n$ form, along the periphery, an everywhere dense set, then the law of continuity permits the conclusion that, for all φ,

$$f(u - 2\pi\bar{\nu},\ \varphi + 2\pi\bar{\nu}) = f(u,\ \varphi). \tag{23.32}$$

Let us construct the function

$$f\left(2\pi\bar{\nu}R\left(\frac{u}{2\pi}\right),\ \varphi - 2\pi\bar{\nu}R\left(\frac{u}{2\pi}\right)\right) = f(u,\ \varphi),$$

possessing the period 2π with respect to φ and u. It is clear that this function is continuous since in view of the identity (23.32), the property of continuity is preserved at the points of discontinuity $R(\frac{u}{2\pi})$.

Let us take, in eq.(23.31),

$$n = \frac{t - t_0}{T} - R\left(\frac{t - t_0}{T}\right).$$

Then,

$$\theta(t) = 2\pi\frac{\bar{\nu}}{T}(t - t_0) + \psi + f\left(2\pi\frac{t - t_0}{T},\quad 2\pi\frac{\bar{\nu}(t - t_0)}{T} + \psi\right).$$

Substituting this expression in eq.(23.13), we can prove that the solutions of the fundamental equation (23.1) under consideration, lying on the integral manifold S, have in this case the form

$$\left.\begin{aligned} x(t) &= \Phi(\alpha_e t,\ \alpha_p t + \psi), \quad \psi = \text{const}, \\ \alpha_e &= \frac{2\pi}{T}, \quad \alpha_p = \frac{2\pi\bar{\nu}}{T}, \end{aligned}\right\} \tag{23.33}$$

where $\Phi(\varphi,\ \psi)$ is an arbitrary function of the angular variables φ, ψ with the period 2π.

Thus, these solutions, lying on the manifold S, are found to be quasi-periodic and have two fundamental frequencies, an "external" frequency $\frac{2\pi}{T}$ and a "natural" frequency $\frac{2\pi\bar{\nu}}{T}$.

We may note that

$$\frac{2\pi\bar{\nu}}{T} = \lim_{t\to\infty} \frac{\theta(t)}{t}.$$

Therefore, in accordance with the inequation (23.26), we have,

$$|\alpha_p - \Omega(\varepsilon)| \leqslant \varepsilon\delta^*(\varepsilon). \tag{23.34}$$

Thus $\Omega(\varepsilon)$ is an asymptotic approximation for the natural part α_p.

Let us consider the case when the number $\bar{\nu}$ is rational. Then, in view of the above result 3) by Poincaré-Denjoy, there will be periodic solutions with a period sT on the integral manifold S.

Any solution belonging to S will approach one of these periodic solutions, as $t \to \infty$.

We note, among other things, that since the frequencies of the periodic solutions will be multiples of $\frac{2\pi}{sT}$, we may represent them by the linear combinations of frequencies:

$$\alpha_e = \frac{2\pi}{T}, \quad \frac{2\pi}{sT} r = \frac{2\pi}{T}\bar{\nu} = \alpha_p.$$

Thus, in this case as well, the stationary solutions (periodic solutions lying on the manifold S) may be formally presented as functions having the two fundamental frequencies α_e and α_p.

Let us then consider the solutions not lying on the manifold S, confining ourselves to the case where all the (n − 1) characteristic exponents have negative real parts. We can show that every solution of the fundamental equation passing through any point of the region U_{σ_0}, approaches (as $t \to +\infty$) one of the stationary solutions, i.e., approaches a quasi-periodic solution in the case of an irrational $\bar{\nu}$ or a periodic solution in the case of a rational $\bar{\nu}$.

For this purpose, as will be clear from the inequalities (23.17) of Theorem II, it is sufficient to prove that if any continuous and differentiable functions $\theta(t)$, in the interval (t_0, ∞), satisfies the inequality

$$\left|\frac{d\theta(t)}{dt} - \varepsilon F(t, \theta(t))\right| \leqslant C_2(\varepsilon)\, e^{-\eta(t-t_0)},$$

then

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$$\theta(t) - \varphi(t) \underset{t \to +\infty}{\to} 0,$$

where $\varphi(t)$ is a solution of the equation

$$\frac{d\varphi}{dt} = \varepsilon F(t, \varphi).$$

In turn, for proving this assertion, it is sufficient to prove that, for any sequence θ_n, for which

$$|\theta_{n+1} - F(\theta_n)| \leqslant Q(\varepsilon) e^{-\varepsilon\gamma_0 n}, \quad \gamma_0 = \gamma T, \tag{23.35}$$

the following relation

$$\theta_n - \varphi_n \to 0, \tag{23.36}$$

will hold, where φ_n satisfies the iteration equation

$$\varphi_{n+1} = F(\varphi_n).$$

Thus, to complete our proof, it remains to consider the sequence θ_n, satisfying the inequality (23.35), and to establish for it the validity of the limiting relation (23.36).

We will now do this.

Take the expression

$$\begin{aligned}\theta_{m+1} \pm Ke^{-\varepsilon\gamma_0(m+1)} - F(\theta_m \pm Ke^{-\varepsilon\gamma_0 m}) &= \\ = \theta_{m+1} - F(\theta_m) \pm Ke^{-\varepsilon\gamma_0 m}(e^{-\varepsilon\gamma_0} - \widetilde{F}'_\theta)\end{aligned}$$

and note that, by virtue of eq.(23.22),

$$1 - \varepsilon\rho(\varepsilon) \leqslant F'_\theta \leqslant 1 + \varepsilon\rho(\varepsilon).$$

We therefore have

$$\left.\begin{aligned}\theta_{m+1} + Ke^{-\varepsilon\gamma_0(m+1)} < F(\theta_m + Ke^{-\varepsilon\gamma_0 m}) + \\ + Q(\varepsilon)e^{-\varepsilon\gamma_0 m} - Ke^{-\varepsilon\gamma_0 m}(1 - \varepsilon\rho(\varepsilon) - e^{-\varepsilon\gamma_0}), \\ \theta_{m+1} - Ke^{-\varepsilon\gamma_0(m+1)} > F(\theta_m - Ke^{-\varepsilon\gamma_0 m}) - \\ - Q(\varepsilon)e^{-\varepsilon\gamma_0 m} + Ke^{-\varepsilon\gamma_0 m}(1 - \varepsilon\rho(\varepsilon) - e^{-\varepsilon\gamma_0}).\end{aligned}\right\} \tag{23.37}$$

Let the number ε_0 in Theorem II be taken so small that

$$1 - \varepsilon\rho(\varepsilon) - e^{-\varepsilon\gamma_0} > 0, \quad 0 < \varepsilon \leqslant \varepsilon_0. \tag{23.38}$$

Then, putting

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$$\left.\begin{aligned} K &= \frac{Q(\varepsilon)}{1-\varepsilon\varphi(\varepsilon)-e^{-\varepsilon\gamma_0}}, \\ \psi'_m &= \theta_m + Ke^{-\varepsilon\gamma_0 m}, \quad \psi''_m = \theta_m - Ke^{-\varepsilon\gamma_0 m} \end{aligned}\right\} \tag{23.39}$$

we find from the inequation (23.37),

$$\psi'_{m+1} < F(\psi'_m); \qquad \psi''_{m+1} > F(\psi''_m). \tag{23.40}$$

Consider the systems of numbers

$$\varphi'_{m,n}; \qquad \varphi''_{m,n},$$

which are, with respect to the subscript m, solutions of the iteration eq.(23.26), under the "initial conditions"

$$\left.\begin{aligned} \varphi'_{m,n} &= \theta_n + Ke^{-\varepsilon\gamma_0 n} = \psi'_n, \\ \varphi''_{m,n} &= \theta_n - Ke^{-\varepsilon\gamma_0 n} = \psi''_n \end{aligned}\right\} m = n. \tag{23.41}$$

Because of the inequalities (23.40), bearing in mind the monotonous increase of the function $F(\theta)$, we conclude that

$$\left.\begin{aligned} \psi'_m &\leqslant \varphi'_{m,n}, \\ \psi''_m &\geqslant \varphi''_{m,n}, \end{aligned}\right\} m = n,\ n+1,\ n+2,\ \ldots \tag{23.42}$$

Thus,

$$\varphi''_{m,n} + Ke^{-\varepsilon\gamma_0 m} \leqslant \theta_m \leqslant \varphi'_{m,n} - Ke^{-\varepsilon\gamma_0 m}. \tag{23.43}$$

Let us first consider the case of the irrational $\bar{\nu}$. Then, on the basis of our earlier statement that $\varphi'_{m,n}$ and $\varphi''_{m,n}$ may be represented respectively in the following form:

$$\left.\begin{aligned} \varphi'_{m,n} &= \bar{\nu}m + \xi_n + E(\bar{\nu}m + \xi_n), \\ \varphi''_{m,n} &= \bar{\nu}m + \eta_n + E(\bar{\nu}m + \eta_n) \end{aligned}\right\} \tag{23.44}$$

Bearing in mind eq.(23.43), we may assert that

$$\xi_n > \eta_n. \tag{23.45}$$

Putting m = n + 1 in eqs.(23.43), we have

$$\begin{aligned} \varphi'_{n+1,n} &\geqslant \theta_{n+1} + Ke^{-\varepsilon\gamma_0(n+1)} = \varphi'_{n+1,n+1}, \\ \varphi''_{n+1,n} &\leqslant \theta_{n+1} - Ke^{-\varepsilon\gamma_0(n+1)} = \varphi''_{n+1,n+1}, \end{aligned}$$

whence
$$\xi_n + E(\bar{\nu} n + \bar{\nu} + \xi_n) \geqslant \xi_{n+1} + E(\bar{\nu} n + \bar{\nu} + \xi_{n+1}),$$
$$\eta_n + E(\bar{\nu} n + \bar{\nu} + \eta_n) \leqslant \eta_{n+1} + E(\bar{\nu} n + \bar{\nu} + \eta_{n+1}).$$

Therefore, in view of the monotonicity of the function $\theta + E(\theta)$, we see that
$$\left.\begin{aligned} \xi_{n+1} &\leqslant \xi_n, \\ \eta_{n+1} &\geqslant \eta_n. \end{aligned}\right\} \tag{23.46}$$

On the other hand, it follows from eqs.(23.41), (23.44) that
$$\xi_n + E(\bar{\nu} n + \xi_n) - \eta_n - E(\bar{\nu} n + \eta_n) = 2Ke^{-\sigma n} \xrightarrow[n \to \infty]{} 0,$$

whence
$$\xi_n - \eta_n \xrightarrow[n \to \infty]{} 0. \tag{23.47}$$

The relations (23.45), (23.46), and (23.47) established above show that ξ_n when falling, and η_n when rising, tend toward a certain common limit $\bar{\psi}$.

On the other hand, eqs.(23.43) and (23.44) yield
$$\bar{\nu} m + \xi_m + E(\bar{\nu} m + \xi_m) \geqslant \varphi_m^{(0)} \geqslant \bar{\nu} m + \eta_m + E(\bar{\nu} m + \eta_m),$$

so that
$$\varphi_m^{(0)} - \{(\bar{\nu} m + \bar{\psi}) + E(\bar{\nu} m + \bar{\psi})\} \xrightarrow[m \to \infty]{} 0,$$

which proves the argument.

Let us pass now to the consideration of the case of a rational $\bar{\nu}$:
$$\bar{\nu} = \frac{r}{s}. \tag{23.48}$$

Let us introduce a function obtained as a result of the s-fold application of the transformation F:
$$\Phi_1(\varphi) = F(\ldots F(\varphi)\ldots) - 2\pi r, \tag{23.49}$$

and note that it will be a continuous and monotonously increasing function of φ. The difference
$$\Phi_1(\varphi) - \varphi, \tag{23.50}$$

will obviously have the period 2π in φ.

Since in the case (23.48) under consideration, the periodic solutions of the iteration equation will satisfy the relation

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$$\varphi_{m+s} - \varphi_m = 2\pi r,$$

then for them

$$\varphi_{ms} = \Phi_s(\varphi_{ms}).$$

Thus it remains for us to prove that

$$\varphi_m - 2\pi rm \to \psi_0, \tag{23.51}$$

where ψ_0 is one of the roots of the equation

$$\psi = \Phi_s(\psi). \tag{23.52}$$

Indeed, let us consider a solution ψ_m of the iteration equation, starting from ψ_0.

Then, of course,

$$\psi_{ms} = \psi_0 + 2\pi rm,$$

Therefore, eq.(23.51) will give

$$\varphi_{ms} - \psi_{ms} \to 0, \quad m \to \infty. \tag{23.53}$$

We have, however,

$$\psi_{ms+1} - F(\psi_{ms}) = 0,$$
$$\varphi_{ms+1} - F(\varphi_{ms}) \to 0, \; m \to \infty,$$

whence it follows from eq.(23.53) that

$$\varphi_{ms+1} - \psi_{ms+1} \to 0, \quad m \to \infty.$$

In a completely analogous manner we may establish that, in general,

$$\varphi_{ms+k} - \psi_{ms+k} \to 0, \quad m \to \infty \quad (k = 0, 1, \ldots, s-1),$$

Hence

$$\varphi_n - \psi_n \to 0, \quad n \to \infty.$$

We proceed now to the proof of the limiting relation (23.53).

Let us take a certain solution φ_m of the iteration equation, starting from φ_0 arbitrarily fixed in the interval $(0, 2\pi)$, and put:

$$\bar{\varphi}_m = \varphi_{ms} - 2\pi mr. \tag{23.54}$$

We then have

$$-\alpha_m \leq \bar{\varphi}_m \leq \beta_m, \tag{23.55}$$

where

$$\alpha_m > 0, \quad \beta_m > 0, \quad \alpha_m + \beta_m < 2\pi. \tag{23.56}$$

On the other hand,

$$\bar{\varphi}_{m+1} = \Phi_1(\bar{\varphi}_m)$$

Therefore, in view of the monotonous increase of the function $\Phi_1(\varphi)$, in the case where the solution φ_m is not periodic, i.e., where the numbers $\bar{\varphi}_m$ are not constant with respect to the subscript m, the sequence

$$\bar{\varphi}_0, \ \bar{\varphi}_1, \ \ldots, \ \bar{\varphi}_m, \ \ldots$$

will, monotonously falling or rising, tend toward a certain limit which is a root of eq.(23.52).

It is now readily possible to establish the following arguments:

a) The sequence

$$\bar{\varphi}_m \quad (m = 0, \ 1, \ 2, \ \ldots)$$

can never "jump" across any root φ_0 of eq.(23.52), i.e., there exists no integer m_0 such that

$$\bar{\varphi}_{m_0} < \varphi_0 < \bar{\varphi}_{m_0+1} \tag{23.57}$$

or

$$\bar{\varphi}_{m_0} > \varphi_0 > \bar{\varphi}_{m_0+1}. \tag{23.58}$$

Indeed, let us assume that the inequation (23.57) applies. Then, because of the monotonicity of the function $\Phi_1(\varphi)$, we have

$$\Phi_1(\bar{\varphi}_{m_0}) < \Phi_1(\varphi_0),$$

i.e.,

$$\bar{\varphi}_{m_0+1} < \varphi_0,$$

which contradicts eq.(23.57). By analogy we also prove the impossibility of the inequation (23.58).

b) If the sequences

$$\bar{\varphi}'_m, \ \bar{\varphi}''_m \quad (m = 0, \ 1, \ 2, \ \ldots), \tag{23.59}$$

corresponding to the solutions of the iteration equation, tend toward different roots of eq.(23.52), then the quantities

$$\bar{\varphi}'_m, \quad \bar{\varphi}''_m$$

will always enclose at least one root of this eq.(23.52).

To prove this, we note that two cases can be imagined: when both these sequences simultaneously increase or decrease, and when one of them increases while the other decreases.

Consider first the case of a simultaneous increase or decrease, and let, for example,

$$\bar{\varphi}'_0 > \bar{\varphi}''_0.$$

Then, obviously, for all m,

$$\bar{\varphi}'_m > \bar{\varphi}''_m,$$

Therefore, passing to the limit, we have

$$\varphi' > \varphi'',$$

where φ', φ'' are the roots of eq.(23.52), which are, respectively, the limits of the sequences $\bar{\varphi}'_m$, $\bar{\varphi}''_m$.

Owing to the monotonous increase of the sequence $\bar{\varphi}''_m$, we obviously obtain:

$$\bar{\varphi}''_m < \varphi''. \tag{23.60}$$

Since the sequence $\bar{\varphi}'_m$, on the basis of Argument a), could not "jump" across φ'', then, for all m,

$$\bar{\varphi}'_m > \varphi''. \tag{23.61}$$

The inequations (23.60), (23.61) so established prove, in the case under consideration, the truth of Argument b).

An analogous argument holds for the case of a simultaneous decrease of the sequences (23.59). If, now, one of these sequences decreases and the other increases, then the signs of the following two expressions

$$\bar{\varphi}'_{m+1} - \bar{\varphi}'_m = \Phi_1(\bar{\varphi}'_m) - \bar{\varphi}'_m,$$
$$\bar{\varphi}''_{m+1} - \bar{\varphi}''_m = \Phi_1(\bar{\varphi}''_m) - \bar{\varphi}''_m$$

will differ so that, on the basis of continuity of eq.(23.52),

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$$\Phi_1(\varphi)-\varphi=0$$

has at least one root in the interval

$$[\bar{\varphi}'_m, \ \bar{\varphi}''_m].$$

Thus Argument b) is completely proved.

Let us now return to the inequations (23.43), putting, for brevity,

$$\left.\begin{aligned} \varphi'_{ms,\,ns}-2\pi rm&=\bar{\varphi}'_{m,\,n}, \\ \varphi''_{ms,\,ns}-2\pi rm&=\bar{\varphi}''_{m,\,n}, \\ \theta_{ms}-2\pi rm&=\bar{\theta}_m. \end{aligned}\right\} \tag{23.62}$$

We then have

$$\begin{gathered} \bar{\varphi}'_{m,\,n}-Ke^{-\epsilon\gamma_0 ns}\geqslant\bar{\theta}_m\geqslant\bar{\varphi}''_{m,\,n}+Ke^{-\epsilon\gamma_0 ns} \\ (m=n,\quad n+1,\quad n+2,\ \ldots), \end{gathered} \tag{23.63}$$

and, from eq.(23.40), we find

$$\left.\begin{aligned} \bar{\varphi}'_{n,\,n}&=\bar{\theta}_n+Ke^{-\epsilon\gamma_0 ns}, \\ \bar{\varphi}''_{n,\,n}&=\bar{\theta}_n-Ke^{-\epsilon\gamma_0 ns}. \end{aligned}\right\} \tag{23.64}$$

If now, for any n, as $m\to\infty$, the sequences

$$\bar{\varphi}'_{m,\,n}, \quad \bar{\varphi}''_{m,\,n} \tag{23.65}$$

approach one and the same root ψ_0 of eq.(23.52), then it follows from eq.(23.63) that

$$\theta_{ms}-2\pi rm=\bar{\theta}_m\underset{m\to\infty}{\longrightarrow}\psi_0,$$

thus proving our theorem.

Conversely, if for any n the sequences (23.65) tend toward different roots, then Argument b) proves that the intervals

$$l_n=[\bar{\varphi}'_{n,\,n}, \ \bar{\varphi}''_{n,\,n}]$$

will always contain at least one root of eq.(23.52).

It is easy to prove, however, that at an arbitrary positive integer k, the intervals

$$l_{n+k}, \quad l_n$$

have a common part.

Indeed, in the opposite case, one of the two following inequations would be obtained:

$$\bar{\varphi}'_{n+k,\,n+k} > \bar{\varphi}''_{n+k,\,n+k} > \bar{\varphi}'_{n,\,n} > \bar{\varphi}''_{n,\,n} \qquad (23.66)$$

or

$$\bar{\varphi}''_{n+k,\,n+k} < \bar{\varphi}'_{n+k,\,n+k} < \bar{\varphi}''_{n,\,n} < \bar{\varphi}'_{n,\,n}. \qquad (23.67)$$

Let us assume first that eq.(23.66) holds. Then, substituting in eq.(23.63)

$$m = n + k,$$

we have, by virtue of eq.(23.64),

$$\bar{\varphi}'_{n+k,\,n} \geqslant \bar{\varphi}'_{n+k,\,n+k},$$

Therefore, it follows from eq.(23.66) that

$$\bar{\varphi}'_{n+k,\,n} > \bar{\varphi}'_{n+k,\,n+k} > \bar{\varphi}''_{n+k,\,n+k} > \bar{\varphi}'_{n,\,n}. \qquad (23.68)$$

Let α^*_{n+k} denote a root of eq.(23.52) lying in the interval l_{n+k}. From equation (23.68) we have

$$\bar{\varphi}'_{n+k,\,n} > \alpha^*_n > \bar{\varphi}'_{n,\,n}$$

which proves that the sequence

$$\bar{\varphi}'_{n,\,n};\quad \bar{\varphi}'_{n+1,\,n},\quad \ldots,\quad \bar{\varphi}'_{n+k,\,n}$$

jumps the root of eq.(23.52), which contradicts Argument a).

By analogy we prove the impossibility of the inequations (23.67).

Thus, the intervals

$$l_{n+k},\quad l_n$$

have parts in common. Since, on the other hand, the length l_n is equal to $2Ke^{-\varepsilon\gamma_0 n}$ and since each of these intervals contains $\bar{\theta}_n$ and roots of eq.(23.52), we finally can prove that

$$\theta_{ns} - 2\pi rn = \bar{\theta}_n \to \psi,$$

where ψ is a root of eq.(23.52). This completes the proof of the arguments.

We will summarize these results in the form of the following theorem:

Theorem III. Let the conditions of theorem II be satisfied for $m = 2$. Also,

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let one of the following two conditions be satisfied: a) the function $X(t, x)$ possesses, with respect to the variable t, a period T independent of x or b) in equations (22.24) the functions $W(t, \varphi, b)$, $B(t, \varphi, b)$ have the form

$$W(t, \varphi, b) = \overline{W}(t, \varphi + \nu t, b),$$
$$B(t, \varphi, b) = \overline{B}(t, \varphi + \nu t, b),$$

the functions $\overline{W}(t, \varphi, b)$, $\overline{B}(t, \varphi, b)$ having in t the period T, which does not depend on φ, b.

Then, for a sufficiently small ε_0, the behavior of the solutions lying on the integral manifold S is characterized by the number $\bar{\nu}$; if $\bar{\nu}$ is irrational, then each of these solutions is a quasi-periodic function of t with two fundamental frequencies:

$$\alpha_e = \frac{2\pi}{T}, \quad \alpha_p = \frac{2\pi}{T}\bar{\nu};$$

If $\bar{\nu}$ is rational, then periodic solutions exist on S with these same fundamental frequencies; any nonperiodic solution, lying on S, approaches one of the periodic solutions as $t \to \infty$.

In the case a):

$$|\alpha_p - \varepsilon\omega| \leqslant \varepsilon\delta(\varepsilon), \qquad \delta(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

In the case b):

$$|\alpha_p - \nu - \varepsilon\omega| \leqslant \varepsilon\delta(\varepsilon).$$

Let, in addition to the conditions already imposed, all above $(n - 1)$ characteristic exponents have negative real parts.

Then any solution passing through any point of the region U_σ approaches, as $t \to +\infty$, one of the stationary solutions (a quasi-periodic solution in the case of an irrational $\bar{\nu}$, or a periodic solution in the case of a rational $\bar{\nu}$).

In conclusion, a number of applications of theorems I, II, and III to the theory of nonlinear oscillations in systems with one degree of freedom will be discussed.

Let us begin with a consideration of the free oscillations characterized by a differential equation of the form of eq.(1.1):

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$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}\right)$$

with a small positive parameter ε.

Then the equation of first approximation for the oscillation amplitude will be (1.24)

$$\frac{da}{dt} = \varepsilon A_1(a),$$

while, on the basis of eq.(1.27),

$$A_1(a) = -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, \ -a\omega\sin\psi)\sin\psi\, d\psi.$$

Next, we will show that theorem I and the corresponding remarks permit a rigorous formulation of the results, as to the properties of periodicity and stability, that were obtained in Chapter I for approximate solutions.

For this purpose let us perform in eq.(1.1) the following substitutions of variables

$$x = a\cos\psi,$$
$$\frac{dx}{dt} = -a\omega\sin\psi.$$

As a result we obtain

$$\left.\begin{aligned} \frac{da}{dt} &= -\frac{\varepsilon}{\omega} f(a\cos\psi, -a\omega\sin\psi)\sin\psi, \\ \frac{d\psi}{dt} &= \omega - \frac{\varepsilon}{\omega a} f(a\cos\psi, -a\omega\sin\psi)\cos\psi, \end{aligned}\right\} \quad (23.69)$$

whence

$$\frac{da}{d\psi} = -\frac{\varepsilon}{\omega^2}\frac{f(a\cos\psi, \ -a\omega\sin\psi)\sin\psi}{1 - \frac{\varepsilon}{a\omega^2} f(a\cos\psi, \ -a\omega\sin\psi)\cos\psi}. \quad (23.70)$$

Thus we reach an equation of the form of (21.1).

The corresponding equation of first approximation (21.2) will be

$$\frac{da}{d\psi} = -\frac{\varepsilon}{2\pi\omega^2}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi,$$

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i.e.,

$$\frac{da}{d\psi} = \frac{\varepsilon}{\omega} A_1(a).$$

Let the equation

$$A_1(a) = 0$$

have the nontrivial solution

$$a = a_0, \qquad a_0 \neq 0,$$

for which

$$A_1'(a_0) \neq 0.$$

Assume likewise that the function f(x, x') on the plane (x, x') is continuous, with its partial derivatives of first order, in the neighborhood of the ellipse

$$x^2 + \frac{x'^2}{\omega^2} = a_0^2. \tag{23.71}$$

Then, the above theorem proves that, for sufficiently small values of ε, the exact equation (23.70) has the periodic solution

$$a = a_0(\psi)$$

with a period of 2π, close to a_0. This solution will be stable in the case

$$A_1'(a_0) < 0 \tag{23.72}$$

and possess the property of attracting nearby solutions.

In the case

$$A_1'(a_0) > 0 \tag{23.73}$$

it is unstable and has the property of repulsion.

Bearing in mind the substitution of variables performed by us, and the second equation of (23.69), we see that the eq.(1.1) under consideration has, for sufficiently small ε, a limit cycle corresponding to a periodic solution close to the ellipse of eq.(23.71). Under the condition (23.72), this limit cycle will be stable, under the condition (23.71) it will be unstable.

These are exactly the conclusions drawn in the approximate theory discussed earlier in this book.

Let us now study the oscillatory systems described by the more general equa-

tion (12.1)

$$\frac{d^2x}{dt^2}+\omega^2 x=\varepsilon f\left(\nu t,\ x,\ \frac{dx}{dt}\right),$$

in which $f(\theta,\ x,\ \frac{dx}{dt})$ is a periodic function of θ with period 2π.

Let us first consider the case of resonance, when

$$\omega^2=\left(\frac{p}{q}\nu\right)^2+\varepsilon\Delta,$$

where p, q are relatively prime numbers.

The corresponding equations of first approximation will be

$$\left.\begin{aligned}\frac{da}{dt}&=\varepsilon A_1(a,\ \vartheta),\\ \frac{d\vartheta}{dt}&=\varepsilon B_1(a,\ \vartheta),\end{aligned}\right\}\qquad(23.74)$$

~~where~~, by virtue of eqs.(13.14) and (13.23), we have

$$\left.\begin{aligned}A_1(a,\ \vartheta)&=-\frac{1}{2\pi p}\int_0^{\frac{2\pi q}{\nu}} f_0\left(a,\ \nu t,\ \frac{p}{q}\nu t+\vartheta\right)\times\\ &\qquad\times\sin\left(\frac{p}{q}\nu t+\vartheta\right)dt,\\ B_1(a,\ \vartheta)&=\frac{\Delta q}{2pa}-\frac{1}{2\pi pa}\int_0^{\frac{2\pi q}{\nu}} f_0\left(a,\ \nu t,\ \frac{p}{q}\nu t+\vartheta\right)\times\\ &\qquad\times\cos\left(\frac{p}{q}\nu t+\vartheta\right)dt.\end{aligned}\right\}\qquad(23.75)$$

To make use of our theorems, let us perform, in the fundamental equation (12.1), the following substitution of variables:

$$\left.\begin{aligned}x&=\xi\cos\frac{p}{q}\nu t+\eta\sin\frac{p}{q}\nu t,\\ \frac{dx}{dt}&=-\xi\frac{p}{q}\nu\sin\frac{p}{q}\nu t+\eta\frac{p}{q}\nu\cos\frac{p}{q}\nu t,\end{aligned}\right\}\qquad(23.76)$$

thus reducing it to a system in the standard form:

$$\left.\begin{aligned}\frac{d\xi}{dt}&=\varepsilon X(t,\ \xi,\ \eta),\\ \frac{d\eta}{dt}&=\varepsilon Y(t,\ \xi,\ \eta),\end{aligned}\right\}\qquad(23.77)$$

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where

$$X(t, \xi, \eta) = -\frac{F(t, \xi, \eta)\sin\frac{p}{q}\nu t}{\frac{p}{q}\nu},$$

$$Y(t, \xi, \eta) = \frac{F(t, \xi, \eta)\cos\frac{p}{q}\nu t}{\frac{p}{q}\nu},$$

$$F(t, \xi, \eta) = f\Big(\nu t,\ \xi\cos\frac{p}{q}\nu t + \eta\sin\frac{p}{q}\nu t,\ -\xi\frac{p}{q}\nu\sin\frac{p}{q}\nu t + \eta\frac{p}{q}\nu\cos\frac{p}{q}\nu t\Big) - \Delta\Big(\xi\cos\frac{p}{q}\nu t + \eta\sin\frac{p}{q}\nu t\Big).$$

It will be clear that the right sides of eq.(23.74) are periodic functions of t with the period $\frac{2\pi q}{\nu}$.

We note likewise that the equations of first approximation, corresponding to the system (23.77)

$$\frac{d\xi}{dt} = \varepsilon\frac{\nu}{2\pi q}\int_0^{\frac{2\pi q}{\nu}} X(t, \xi, \eta)\,dt,$$

$$\frac{d\eta}{dt} = \varepsilon\frac{\nu}{2\pi q}\int_0^{\frac{2\pi q}{\nu}} Y(t, \xi, \eta)\,dt,$$

are equivalent to eqs.(23.74) and are transformed into them by means of the substitution

$$\xi = a\cos\vartheta, \quad \eta = -a\sin\vartheta.$$

Assume that the eqs.(23.74) have the constant solution

$$a = a_0, \quad \vartheta = \vartheta_0 \tag{23.78}$$

and that, in the neighborhood of the ellipse

$$x^2 + \frac{x'^2}{\left(\frac{p}{q}\nu\right)^2} = a_0^2 \tag{23.79}$$

the function f(t, x, x') is continuous, together with its partial derivatives of

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first order with respect to x, x'.

Further, let both roots of the characteristic equation corresponding to the equations of variation for the solution (23.78) have real negative parts.

Then, obviously, the conditions of theorem I are satisfied.

Bearing in mind the properties established in it, it may be asserted that, in the case under consideration and for sufficiently small values of ε, eq.(12.1) has a periodic solution with the period $\frac{2\pi q}{\nu}$, close to harmonic:

$$a_0 \cos\left(\frac{p}{q}\nu t + \varphi_0\right).$$

Any solution passing through a point of some neighborhood of the ellipse (23.79) will asymptotically approach the periodic solution as $t \to +\infty$.

Let now the equations of first approximations have a periodic solution with a characteristic exponent having a non-zero real part, and let the function $f(t,x,x')$, in a certain neighborhood of the orbit of this solution, have continuous partial derivatives with respect to x, x', to the second order inclusive.

In this case the conditions of theorem II and theorem III (with the condition a) are satisfied.

For this reason we may assert, for example, that for sufficiently small values of ε, in a certain neighborhood of that orbit, there are stationary solutions having two fundamental frequencies: "natural" and "forced". When the ratio between these frequencies is irrational, the solutions are quasi-periodic; when they are rational, the solutions are periodic.

In the case when the real part of the characteristic exponent is positive, the stationary solutions are unstable. If, on the other hand, this real part is negative, then every solution of eq.(12.1), for which, at any point t_0,

$$\xi(t) = x(t)\cos\frac{p}{q}\nu t - \frac{x'(t)}{\frac{p}{q}\nu}\sin\frac{p}{q}\nu t, \quad \eta(t) = x(t)\sin\frac{p}{q}\nu t + \frac{y'(t)}{\frac{p}{q}\nu}\cos\frac{p}{q}\nu t$$

lies rather close alongside that orbit, asymptotically approaching the stationary state, for $t \to +\infty$.

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Let us now consider the nonresonant case, when the equations of first approximation have the form

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a), \\ \frac{d\psi}{dt} &= \omega + \varepsilon B_1(a), \end{aligned}\right\} \qquad (23.80)$$

where $A_1(a)$ and $B_1(a)$ are determined by eqs.(12.15), (12.35):

$$A_1(a) = -\frac{1}{4\pi^2\omega}\int_0^{2\pi}\int_0^{2\pi} f(\theta,\ a\cos\psi,\ -a\omega\sin\psi)\sin\psi\,d\psi\,d\theta,$$

$$B_1(a) = -\frac{1}{4\pi^2\omega a}\int_0^{2\pi}\int_0^{2\pi} f(\theta,\ a\cos\psi,\ -a\omega\sin\psi)\cos\psi\,d\psi\,d\theta.$$

For convenience in applying theorems II and III, let us perform in eq.(12.1) the following substitution of variables:

$$x = a\cos(\omega t + \varphi),$$

$$\frac{dx}{dt} = -a\omega\sin(\omega t + \varphi).$$

We obtain the equation in standard form:

$$\left.\begin{aligned} \frac{da}{dt} &= \varepsilon A(\nu t,\ \omega t + \varphi,\ a), \\ \frac{d\varphi}{dt} &= \varepsilon B(\nu t,\ \omega t + \varphi,\ a), \end{aligned}\right\} \qquad (23.81)$$

where

$$A(\theta,\ \psi,\ a) = -\frac{1}{\omega} f(\theta,\ a\cos\psi,\ -a\omega\sin\psi)\sin\psi,$$

$$B(\theta,\ \psi,\ a) = -\frac{1}{\omega a} f(\theta,\ a\cos\psi,\ -a\omega\sin\psi)\cos\psi.$$

The right sides of these equations obviously are quasi-periodic functions of t with two fundamental frequencies, ω and ν.

If the ratio of these frequencies is irrational, we have:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T A(\nu t,\ \omega t + \varphi,\ a)\,dt =$$

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$$\left.\begin{aligned} &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} A(\theta,\psi,a)\,d\theta\,d\psi = A_1(a),\\ \lim_{T\to\infty}\frac{1}{T}\int_0^T &B(\nu t,\omega t+\varphi,a)\,dt =\\ &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} B(\theta,\psi,a)\,d\theta\,d\psi = B_1(a). \end{aligned}\right\} \qquad (23.82)$$

We note that these equations can be satisfied, even when the value of the ratio $\frac{\omega}{\nu}$ is not irrational.

Take, for example the case considered in Chapter III, where $f(\nu t, x, x')$ is represented by a finite sum of the form

$$\sum_{-N}^{N} e^{in\nu t} f_n(x, x'),$$

in which $f_n(x, x')$ are polynomials in x, x'.

As is easy to see, in this case a finite aggregate of rational numbers can be assigned such that, if $\frac{\omega}{\nu}$ is not equal to one of the numbers of this aggregate, then eq.(23.82) is satisfied.

Let the validity of these equations be proved, in this or any other manner.

Then, eqs.(23.80) will be the equations of first approximation (averaged equations) for the system (23.81).

Assume that the equation

$$A_1(a) = 0$$

has the nontrivial solution

$$a = a_0 \neq 0,$$

for which

$$A_1'(a\ ) \neq 0.$$

Assume also that the function $f(\theta, x, x')$ has continuous partial derivatives with respect to x, x' to second order inclusive in a certain neighborhood of the ellipse

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$$x^2 + \frac{x'^2}{\omega^2} = a_0^2.$$

In this case, the conditions of theorems II and theorem III [condition b)] will obviously be satisfied.

Consequently we may assert that, for sufficiently small values of ε, eq.(12.1) actually does have stationary solutions of amplitude close to a_0, which solutions, as functions of t, have two fundamental frequencies, a natural and a forced.

For

$$A_1'(a_0) > 0$$

the family of stationary solutions has the property of repulsion, while for

$$A_1'(a_0) < 0$$

it has the property of attraction of nearby solutions.

It is interesting to note that the conditions in question here, making theorems II and III applicable, are so general that, when they are satisfied, even the series (12.46), which enters into the refined first approximation, may still be divergent*.

We have considered the questions of establishing the properties of exact solutions from the properties of the solutions of the equations of first approximation.

In a number of cases, however, it may be of interest to use equations of a higher approximation for this purpose.

Thus, for example, the real parts of the characteristic exponents may vanish for equations of the first approximation.

The question of the theoretical evaluation of error for an asymptotic approximation of higher order may also arise.

For such cases it is easy to generalize the technique described in Section 22, for example, using the expression of the refined m^{th} approximation as the formulas of substitution of variables. In that case we arrive at a system of the type of eq.(22.61) in which the "additional terms" P, Q will already be quantities of the

* Due to the presence of "small divisors" of the form $\omega^2 - (n\alpha + m\omega)^2$.

order of smallness of ε^{m+1}, which, of course, correspondingly raises the order of the evaluations so obtained.

Naturally such a consideration requires the imposition of more severe conditions on the character of the regularity of the functions entering into the differential equations under investigation.

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ERRATA SHEET*

(* Translator's Note: All corrections of errata sheet have been made in text)

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